IV. *On Simultaneous Partial Differential Equations.*

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Contents.

§ 1. In this paper, without touching on the question of the existence of integrals of systems of simultaneous partial differential equations, I have given a method by which the problem of finding their complete primitives may be attacked.

The cases discussed are two: that of a pair of equations of the first order in two dependent and two independent variables, and that of a single equation of the second order, with one dependent and two independent variables.

I follow, as far as possible, the analogy of the method of Lagrange and Charpit, and with this object introduce the conception of the "bidifferential" or differential element of the second order, which bears the same relation to a Jacobian taken with respect to two independent variables as a differential does to a differential coefficient.

The solutions considered are, in general, complete primitives, that is, such as contain arbitrary constants in such number that the result of their elimination is the system of equations proposed for solution. The existence of such primitives is sufficiently established (see the papers of Frau von Kowalevsky and Professor Königsberger, quoted hereafter); it will therefore be assumed, and the object of the investigation

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will be to find conditions that must be satisfied by the equations of the solution and to put these conditions in a convenient form for solution by inspection.

I should add that I am greatly indebted to the referees for their suggestions and for help in removing obscurities.

To the list of authorities given by Dr. FORSYTH (' Theory of Differential Equations,' Part I., pp. 299, 331), may be added the following :—

JULIUS KÖNIG. Math. Annalen, vol. 23, pp. 520, 521.

LEO KÖNIGSBERGER. Crelle, vol. 109, pp. 261–340.* Math. Annalen, vol. 41, pp. 260–285.† Math. Annalen, vol. 44, pp. 17–40.

ED. V. WEBER. München Ber., vol. 25, 423–442.

J. M'COWAN. Edinb. Math. Soc. Proc., vol. 10, 63–70.

HAMBURGER. Crelle, vol. 110, pp. 158–176.

C. BOURLET. Annales de l'École Normale (3), vol. 8.

RIQUIER. Comptes Rendus, vols. 114, 116, 119. Annales de l'École Normale (3), vol. 10.

LLOYD TANNER. Proc. Lond. Math. Soc., vols. 7–11.

J. BRILL. Quarterly Journal of Math., vol. 30, pp. 221–242.

Several of the above papers are only known to me through abstracts.

On Bidifferentials.

§ 2. The idea of a "complete differential" plays an important part in the theory of differential equations. In this paper I shall try to show the importance of an extension of the same idea to differential elements of higher orders, such as enter into multiple integrals.

An expression $\mathrm{X}dx + \mathrm{Y}dy$ is called a complete differential when X, Y are functions of the independent variables x, y, such that

$$\partial\mathrm{Y}/\partial x = \partial\mathrm{X}/\partial y.$$

If this is the case, then, under certain restrictions, the value of $\int(\mathrm{X}dx + \mathrm{Y}dy)$ depends only on the limiting values of the variables, and not on the intermediate ones by which these limits are connected, or, as generally expressed, on the path along which the integral is taken.

This depends on the theorem that

$$\int(\mathrm{X}dx + \mathrm{Y}dy) = \iint\left(\frac{\partial\mathrm{Y}}{\partial x} - \frac{\partial\mathrm{X}}{\partial y}\right)dxdy$$

* For reasons stated below, I am not in agreement with the results given in the latter part of this paper.

† In this paper it should be noticed that the equations (52) on p. 266 are not more general than (46).

when the single integral is taken round the boundary of the area over which the double integral is to extend.

Further, X, Y are in this case the partial derivatives of a single function.

§ 3. Let us consider the double integral

$$\iint(\mathrm{X}\,dy\,dz + \mathrm{Y}\,dz\,dx + \mathrm{Z}\,dx\,dy),$$

where X, Y, Z are functions of the independent variables x, y, z. It is known that this, taken over a closed surface under certain restrictions, is equal to the triple integral

$$\iiint(\partial\mathrm{X}/\partial x + \partial\mathrm{Y}/\partial y + \partial\mathrm{Z}/dz)\,dx\,dy\,dz$$

taken over the space enclosed by that surface.

Hence, if $\partial\mathrm{X}/\partial x + \partial\mathrm{Y}/\partial y + \partial\mathrm{Z}/\partial z = 0$ identically the double integral taken over a closed surface vanishes, and taken over two open surfaces with the same boundary has the same value; that is to say, the value of the double integral depends on the values of x, y, z at the boundary only, and not, under certain restrictions, on the form of the surface enclosed by the boundary.

By analogy we may call the element of the double integral a "complete double differential," or a "complete bidifferential" under these circumstances; the condition that $\mathrm{X}\,dy\,dz + \mathrm{Y}\,dz\,dx + \mathrm{Z}\,dx\,dy$ may be a complete bidifferential is thus

$$\partial\mathrm{X}/\partial x + \partial\mathrm{Y}/\partial y + \partial\mathrm{Z}/\partial z = 0.$$

§ 4. A complete bidifferential may be expressed as a single term, such as $du\,dv$. For let u, v be two independent solutions of the equation

$$\mathrm{X}\frac{\partial u}{\partial x} + \mathrm{Y}\frac{\partial u}{\partial y} + \mathrm{Z}\frac{\partial u}{\partial z} = 0,$$

so that $u = a$, $v = b$ are integrals of the system

$$dx/\mathrm{X} = dy/\mathrm{Y} = dz/\mathrm{Z}\,;$$

then
$$\mathrm{X} = \theta\frac{\partial(u, v)}{\partial(y, z)}, \quad \mathrm{Y} = \theta\frac{\partial(u, v)}{\partial(z, x)}, \quad \mathrm{Z} = \theta\frac{\partial(u, v)}{\partial(x, y)},$$

θ being some multiplier,

and
$$\frac{\partial\mathrm{X}}{\partial x} + \frac{\partial\mathrm{Y}}{\partial y} + \frac{\partial\mathrm{Z}}{\partial z} = \frac{\partial(\theta, u, v)}{\partial(x, y, z)}.$$

Since the last vanishes identically θ is a function of u, v only; a function w of u, v may be found, such that $\partial w/\partial u = \theta$, and thus

$$\mathrm{X} = \frac{\partial(w, v)}{\partial(y, z)}, \quad \mathrm{Y} = \frac{\partial(w, v)}{\partial(z, x)}, \quad \mathrm{Z} = \frac{\partial(w, v)}{\partial(x, y)}.$$

Now in finding the value of the double integral taken over a part of any surface, it will be natural to suppose the co-ordinates of any point of such a surface to be functions of two parameters, say p, q, and to transform the integral into one taken with respect to these. The integral as transformed is

$$\iint\left\{X\frac{\partial(y, z)}{\partial(p, q)}+Y\frac{\partial(z, x)}{\partial(p, q)}+Z\frac{\partial(x, y)}{\partial(p, q)}\right\}dp\,dq,$$

and the known values in terms of p, q are to be substituted for x, y, z and their derivatives.

The subject of integration is

$$\frac{\partial(w, v)}{\partial(y, z)}\frac{\partial(y, z)}{\partial(p, q)}+\frac{\partial(w, v)}{\partial(z, x)}\frac{\partial(z, x)}{\partial(p, q)}+\frac{\partial(w, v)}{\partial(x, y)}\frac{\partial(x, y)}{\partial(p, q)}, \quad \text{or}$$

$$\begin{vmatrix} \dfrac{\partial w}{\partial x}\dfrac{\partial x}{\partial p}+\dfrac{\partial w}{\partial y}\dfrac{\partial y}{\partial p}+\dfrac{\partial w}{\partial z}\dfrac{\partial z}{\partial p}, & \dfrac{\partial v}{\partial x}\dfrac{\partial x}{\partial p}+\dfrac{\partial v}{\partial y}\dfrac{\partial y}{\partial p}+\dfrac{\partial v}{\partial z}\dfrac{\partial z}{\partial p} \\ \dfrac{\partial w}{\partial x}\dfrac{\partial x}{\partial q}+\dfrac{\partial w}{dy}\dfrac{\partial y}{\partial q}+\dfrac{\partial w}{dz}\dfrac{\partial z}{\partial q}, & \dfrac{\partial v}{dx}\dfrac{\partial x}{\partial q}+\dfrac{\partial v}{\partial y}\dfrac{\partial y}{\partial q}+\dfrac{\partial v}{\partial z}\dfrac{\partial z}{\partial q} \end{vmatrix} \quad \text{or}$$

$$\frac{\partial(w, v)}{\partial(p, q)}.$$

The integral is therefore

$$\iint\frac{\partial(w, v)}{\partial(p, q)}\,dp\,dq, \quad \text{or} \quad \iint dw\,dv,$$

and if we take a single element we may write

$$X\,dy\,dz+Y\,dz\,dx+Z\,dx\,dy=dw\,dv,$$

dropping the parameters p, q, since the values which x, y, z have in terms of them are immaterial.

This equation is meaningless unless the expression in terms of parameters is understood. The same is true of ordinary differentials. If when u is a function of x, y, z we write

$$du=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy+\frac{\partial u}{\partial z}dz,$$

we mean that if x, y, z are supposed to be any functions whatever of a single parameter p, then

$$\frac{du}{dp}=\frac{\partial u}{\partial x}\frac{dx}{dp}+\frac{\partial u}{\partial y}\frac{dy}{dp}+\frac{\partial u}{\partial z}\frac{dz}{dp}.$$

This equation being true quite independently of the expressions assumed for x, y, z in terms of p, we drop the denominator dp for convenience; but in modern works on the Differential Calculus it is quite understood that a differential by itself is meaningless apart from this or some equivalent convention.

§ 5. The functions w, v are not uniquely determined. They may be replaced by W, V, where W, V are functions of w, v, one of which, say W, is arbitrary, while V is only restricted by the condition

$$\frac{\partial(\mathrm{W}, \mathrm{V})}{\partial(w, v)} = 1.$$

The transformations of w, v which are allowable will thus form a group. For a single integral the operations of the corresponding group consist in the addition of different constants, that is, in varying the constant of integration; the theory of periodic functions is connected with discontinuous sub-groups of this. It is possible that an investigation of the discontinuous sub-groups of the group of transformations of two variables which leaves their bidifferential unchanged may lead to an extended theory of periodic functions of the two variables.

§ 6. The finding of the functions w, v may be considered as the indefinite integration of the bidifferential expression. It is simplified by Jacobi's theory of the last multiplier, which is here a constant.

Since
$$\mathrm{X} = \frac{\partial(w, v)}{\partial(y, z)}, \quad \mathrm{Y} = \frac{\partial(w, v)}{\partial(z, x)}$$

we have
$$\mathrm{X}dy - \mathrm{Y}dx = \frac{\partial v}{\partial z} dw - \frac{\partial w}{\partial z} dv ;$$

and thus, on the supposition that v is constant,

$$dw = \frac{\mathrm{X}\,dy - \mathrm{Y}\,dx}{\frac{\partial v}{\partial z}} = \frac{\mathrm{Y}\,dz - \mathrm{Z}\,dy}{\frac{\partial v}{\partial x}} = \frac{\mathrm{Z}\,dx - \mathrm{X}\,dz}{\frac{\partial v}{xy}}$$

$$= \frac{(\mu \mathrm{Z} - \nu \mathrm{Y})\,dx + (\nu \mathrm{X} - \lambda \mathrm{Z})\,dy + (\lambda \mathrm{Y} - \mu \mathrm{X})\,dz}{\lambda \frac{\partial v}{\partial x} + \mu \frac{\partial v}{\partial y} + \nu \frac{\partial v}{\partial z}}.$$

Hence w may be found, if v is known, by integrating this last expression on the supposition that v is constant; λ, μ, ν may have any values and the constant of integration is to be replaced by an arbitrary function of v. Thus, when one of the functions w, v is known, the other is found by ordinary integration. The only restriction on the one found first is the equation

$$\mathrm{X}\frac{\partial v}{\partial x} + \mathrm{Y}\frac{\partial v}{\partial y} + \mathrm{Z}\frac{\partial v}{\partial z} = 0.$$

§ 7. Let us now suppose a greater number of independent variables. Let u be a function of $x_1, x_2 \; . \; . \; . \; x_n$.

We have the relation

$$du = \frac{\partial u}{\partial x_1} dx_1 + \frac{\partial u}{\partial x_2} dx_2 + \; . \; . \; . + \frac{\partial u}{\partial x_n} dx_n.$$

Here the differentials represent simultaneous infinitesimal increments, those of the independent variables being arbitrary. The equation may also be interpreted by supposing $x_1, x_2 \ldots x_n$ to depend in any manner on a single parameter p, when the equation

$$\frac{du}{dp} = \sum_{r=1}^{n} \frac{\partial u}{\partial x_r} \frac{dx_r}{dp}$$

holds whatever functions of the parameter we suppose $x_1 \ldots x_n$ to be.

To get the idea of a double differential we must suppose two sets of simultaneous infinitesimal increments; denote them by d, δ. The bidifferential of x, y is then $dx \,.\, \delta y - \delta x \,.\, dy$.* This vanishes if x, y are not functionally independent, just as dx vanishes if x is a constant. The analogy is very clearly shown if we say that dx vanishes when some function $\phi(x)$ vanishes, $dx\, dy$ vanishes when some function $\phi(x, y)$ vanishes.

If u, v are functions of n independent variables $x_1, x_2 \ldots x_n$, we have

$$du = \sum_{r=1}^{n} \frac{\partial u}{\partial x_r} dx_r, \qquad \delta u = \sum_{r=1}^{n} \frac{\partial u}{\partial x_r} \delta x_r$$

$$dv = \sum_{r=1}^{n} \frac{\partial v}{\partial x_r} dx_r, \qquad \delta v = \sum_{r=1}^{n} \frac{\partial v}{\partial x_r} \delta x_r, \text{ and hence}$$

$$du \,.\, \delta v - \delta u \,.\, dv = \sum_{r=1}^{n} \sum_{s=1}^{n} \frac{\partial u}{\partial x_r} \frac{\partial v}{\partial x_s} (dx_r \,.\, \delta x_s - dx_s \,.\, \delta x_r),$$

or

$$du\, dv = \Sigma \frac{\partial(u, v)}{\partial(x_r, x_s)} dx_r\, dx_s$$

the summation being taken over all pairs of different suffixes r, s. Hence the expression for $du\, dv$ is formed by multiplying together

$$\sum_{r=1}^{n} \frac{\partial u}{\partial x_r} dx_r \quad \text{and} \quad \sum_{r=1}^{n} \frac{\partial v}{\partial x_r} dx_r$$

with the conventions

$$dx\, dy = - dy\, dx,$$
$$dx\, dx = 0.$$

We shall often use the notation $d(x, y)$ for $dx\, dy$.

§ 8. For the purpose of double integration of such an expression as $\sum_{r,s} X_{rs}\, d(x_r, x_s)$, in which the coefficients X are functions of $x_1 \ldots x_n$, it is natural to suppose $x_1 \ldots x_n$ expressed throughout the range of the integration in terms of two parameters, say p, q. The integral thus becomes

* The dot is used here and throughout the paragraph to distinguish multiplication in the ordinary algebraic sense from multiplication according to the Grassmann conventions stated at the end of the paragraph.

$$\iint \sum_{r,s} X_{rs} \frac{d(x_r, x_s)}{d(p, q)} dp\, dq.$$

If $X_{rs} = \dfrac{\partial(u, v)}{\partial(x_r, x_s)}$ for all pairs of suffixes, the subject of integration in the last integral is $d(u, v)/d(p, q)$, so that the integral becomes $\iint du\, dv$. Its value will therefore only depend on the values of u, v, that is of $x_1, x_2 \ldots x_n$, at the boundary of the range of integration, and not on the form of the relations giving $x_1, x_2 \ldots$ in terms of p, q, which define the particular surface over which the integral is taken.

In this case we may write

$$\sum_{r,s} X_{rs}\, d(x_r, x_s) = d(u, v)$$

and call it a complete bidifferential. It is easily seen that the coefficients X satisfy the relations

$$X_{rs} X_{ij} + X_{ri} X_{js} + X_{rj} X_{si} = 0, \quad \ldots \ldots \ldots \quad (1)$$

$$\frac{\partial X_{rs}}{\partial x_i} + \frac{\partial X_{ir}}{\partial x_s} + \frac{\partial X_{si}}{\partial x_r} = 0 \quad \ldots \ldots \ldots \ldots \quad (2),$$

for all combinations of suffixes, where it is understood that the term $X_{rs} d(x_r, x_s)$ may be also written $X_{sr} d(x_s, x_r)$, so that

$$X_{rs} = - X_{rs}.$$

The conditions (2) are those which must be satisfied in order that the value of the double integral may depend only on the boundary. The difference of two values of the double integral, for which the same boundary is assumed, will be its value over a closed surface passing through the boundary curve, and this may be transformed into the triple integral

$$\iiint \sum_{r,s,i} \left(\frac{\partial X_{rs}}{\partial x_i} + \frac{\partial X_{ir}}{\partial x_s} + \frac{\partial X_{si}}{\partial x_r}\right) dx_i\, dx_r\, dx_s$$

taken through the volume of any solid bounded by this closed surface. Hence this integral must vanish for any solid. By taking an infinitesimal solid, for every point of which all but x_i, x_r, x_s are constant, we find the condition (2).

The conditions (2) would be satisfied by an expression which was the sum of two or more complete bidifferentials, but (1) in general would not.

§ 9. We next try to find whether these conditions are sufficient as well as necessary. Now all the coefficients X cannot vanish. Suppose that X_{12} does not, then we have from (1)

$$X_{rs} = \frac{X_{1r} X_{2s} - X_{1s} X_{2r}}{X_{12}} \quad (r, s = 3, 4 \quad \ldots \quad n)$$

and in virtue of these all the conditions (1) are satisfied.

Taking the values thus given for X_{rs}, X_{ir}, X_{si} we have

$$(rsi) \equiv \frac{\partial X_{rs}}{\partial x_i} + \frac{\partial X_{ir}}{\partial x_s} + \frac{\delta X_{si}}{\partial x_r}$$

$$= \frac{X_{2s}}{X_{12}}\left(\frac{\partial X_{1r}}{\partial x_i} + \frac{\partial X_{i1}}{\partial x_r}\right) + \frac{X_{1s}}{X_{12}}\left(\frac{\partial X_{r2}}{\partial x_i} + \frac{\partial X_{2i}}{\partial x_r}\right) + \frac{X_{1r}}{X_{12}}\left(\frac{\partial X_{2s}}{\partial x_i} + \frac{\partial X_{i2}}{\partial x_s}\right)$$

$$+ \frac{X_{2r}}{X_{12}}\left(\frac{\partial X_{s1}}{\partial x_i} + \frac{\partial X_{1i}}{\partial x_s}\right) + \frac{X_{1i}}{X_{12}}\left(\frac{\partial X_{s2}}{\partial x_r} + \frac{\partial X_{2r}}{\partial x_s}\right) + \frac{X_{2i}}{X_{12}}\left(\frac{\partial X_{1s}}{\partial x_r} + \frac{\partial X_{r1}}{\partial x_s}\right)$$

$$- \frac{X_{rs}}{X_{12}}\frac{\partial X_{12}}{\partial x_i} - \frac{X_{ir}}{X_{12}}\frac{\partial X_{12}}{\partial x_s} - \frac{X_{si}}{X_{12}}\frac{\partial X_{12}}{\partial x_r}$$

$$= \frac{X_{2s}}{X_{12}}(1ri) + \frac{X_{1s}}{X_{12}}(r2i) + \frac{X_{1r}}{X_{12}}(2si) + \frac{X_{2r}}{X_{12}}(s1i) + \frac{X_{1i}}{X_{12}}(s2r) + \frac{X_{2i}}{X_{12}}(1sr)$$

$$+ \frac{1}{X_{12}}\frac{\partial}{\partial x_1}(X_{ir}X_{2s} + X_{si}X_{2r} + X_{rs}X_{2i}) + \frac{1}{X_{12}}\frac{\partial}{\partial x_2}(X_{ri}X_{1s} + X_{is}X_{1r} + X_{sr}X_{1i})$$

$$+ \frac{X_{ir}}{X_{12}}(21s) + \frac{X_{si}}{X_{12}}(21r) + \frac{X_{rs}}{X_{12}}(21i).$$

Thus the conditions (2) are not independent, but all follow from those in which at least one of the suffixes 1, 2 enters. If they are satisfied then the equations

$$\sum_{r=2}^{n} X_{1r}\,dx_r = 0,\ \sum_{r=1}^{n} X_{2r}\,dx_r = 0$$

can be satisfied by two integrals of the form $u = a$, $v = b$; that is, these last equations will give x_1, x_2 as functions of the rest, such that

$$\frac{\partial x_1}{\partial x_r} = \frac{X_{2r}}{X_{12}},\ \frac{\partial x_2}{\partial x_r} = \frac{X_{r1}}{X_{12}}.$$

For the conditions necessary and sufficient* for this are the vanishing of such expressions as

$$\frac{\partial}{\partial x_s}\frac{X_{r1}}{X_{12}} + \frac{X_{2s}}{X_{12}}\frac{\partial}{\partial x_1}\frac{X_{r1}}{X_{12}} + \frac{X_{s1}}{X_{12}}\frac{\partial}{\partial x_2}\frac{X_{r1}}{X_{12}} - \frac{\partial}{\partial x_r}\frac{X_{s1}}{X_{12}} - \frac{X_{2r}}{X_{12}}\frac{\partial}{\partial x_1}\frac{X_{s1}}{X_{12}} - \frac{X_{r1}}{X_{12}}\frac{\partial}{\partial x_2}\frac{X_{s1}}{X_{12}}$$

in which 1, 2 may be interchanged and r, s are any two of the other suffixes. This expression may be written

$$\frac{1}{X_{12}}(r1s) - \frac{X_{r1}}{X_{12}^2}(12s) + \frac{X_{s1}}{X_{12}^2}(12r) + \frac{1}{X_{12}^2}\frac{\partial}{\partial x_1}(X_{12}X_{rs} + X_{r1}X_{2s} + X_{2r}X_{1s}),$$

so that it vanishes and the conditions of integrability of the equations $\sum_r X_{1r}dx_r = 0$, $\sum_r X_{2r}dx_r = 0$ are satisfied. If $u = a$, $v = b$ are the integrals, then, since $\sum_r X_{1r}dx_r$ does not contain the differential of x_1, we must have

* For proof of this statement see FORSYTH, 'Theory,' part I., pp. 43–51.

$$\sum_r X_{1r}\, dx_r = \vartheta \left\{ \frac{\partial u}{\partial x_1} dv - \frac{\partial v}{\partial x_1} du \right\}$$

and
$$X_{1r} = \theta \frac{\partial(u,\ v)}{\partial(x_1,\ x_r)}.$$

In like manner
$$X_{2r} = \theta \frac{\partial(u,\ v)}{\partial(x_2,\ x_r)},$$
the multiplier θ being the same.

Hence
$$X_{rs} = \theta \frac{\partial(u,\ v)}{\partial(x_r,\ x_s)}$$

and
$$(rst) = \frac{\partial(\theta,\ u,\ v)}{\partial(x_r,\ x_s,\ x_t)}.$$

Since this vanishes for all combinations of suffixes, θ is a function of u, v, and if another function of them, w, is so chosen that

$$\partial w/\partial u = \theta,$$

we shall have
$$\sum_{r,\ s} X_{rs}\, d(x_r,\ x_s) = \theta\, d(u,\ v) = d(w,\ v).$$

Linear Differential Equations.

§ 10. If $u = a$ is an integral of the linear partial differential equation

$$X_1 \frac{\partial x_{n+1}}{\partial x_1} + X_2 \frac{\partial x_{n+1}}{\partial x_2} + \ldots + X_n \frac{\partial x_{n+1}}{\partial x_n} = X_{n+1},$$

where $X_1,\ X_2 \ldots X_{n+1}$ are functions of $x_1, \ldots x_{n+1}$, then u satisfies the condition

$$\sum_{r=1}^{n+1} X_r \frac{\partial u}{\partial x_r} = 0,$$

and the complete differential du is a linear combination* of the determinants

$$\left\| \begin{matrix} dx_1, & dx_2, & dx_3 & \ldots & dx_n, & dx_{n+1} \\ X_1, & X_2, & X_3 & \ldots & X_n, & X_{n+1} \end{matrix} \right\|$$

the coefficients in the combination being usually functions of $x_1, \ldots x_{n+1}$.

If $u = a$ is a common solution of the above equation and of

$$X_1' \frac{\partial x_{n+1}}{\partial x_1} + \ldots + X_n' \frac{\partial x_{n+1}}{\partial x_n} = X'_{n+1},$$

then, in like manner, du is a linear combination of the determinants

* This is generally expressed by saying that "$u = a$ is an integral of the equations

$$\frac{dx_1}{X_1} = \frac{dx_2}{X_2} = \ldots = \frac{dx_{n+1}}{X_{n+1}}."$$

For the sake of the analogy with the work of § 11, I prefer the phrase in the text, which expresses no more and no less than the one generally used.

$$\left\| \begin{matrix} dx_1, & dx_2, & \ldots, & dx_{n+1}, \\ X_1, & X_2, & \ldots, & X_{n+1}, \\ X_1', & X_2', & \ldots, & X'_{n+1}, \end{matrix} \right\|$$

but in general, of course, it will not be possible to combine them so as to form a perfect differential.

§ 11. An analogous process of integration may be given for two simultaneous equations

$$\left.\begin{aligned} \sum_{i,j}\{A_{ij}(p_iq_j - p_jq_i)\} + \sum_i B_ip_i + \sum_i C_iq_i + E = 0 \\ \sum_{i,j}\{A'_{ij}(p_iq_j - p_jq_i)\} + \sum_i B'_ip_i + \sum_i C'_iq_i + E' = 0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (3),$$

in which the coefficients A, B, C, E, A′, B′, C′, E′ are functions of n independent variables, $x_1, x_2 \ldots x_n$, and two dependent y, z, and

$$p_i = \frac{\partial y}{\partial x_i}, \quad q_i = \frac{\partial z}{\partial x_i}.$$

To fix the ideas, take $n = 3$ and let x_4, x_5 stand for y, z respectively, A_{i4} for C_i, A_{i5} for $-B_i$, A_{45} for E, and make similar changes in the accented letters. Then, if $u = a$, $v = b$ are two equations constituting a solution,* a, b being arbitrary constants, we must have

$$\left.\begin{aligned} \sum_{i,j=1,2\ldots5} A_{ij}\frac{\partial(u, v)}{\partial(x_i, x_j)} = 0 \\ \sum_{i,j} A'_{ij}\frac{\partial(u, v)}{\partial(x_i, x_j)} = 0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4);$$

for

$$p_1\frac{\partial(u, v)}{\partial(y, z)} + \frac{\partial(u, v)}{\partial(x_1, z)} = 0, \text{ \&c., if } u = a, \ v = b,$$

and the values thus given for $p_1, q_1, p_2, q_2, p_3, q_3$ must satisfy the equations (3) identically, since a, b are supposed arbitrary. The equations to be solved are thus reduced to others which are linear and homogeneous in the Jacobians, and which do not contain the dependent variables.

The equations (4) give two of the Jacobians of u, v linearly in terms of the others; if we substitute for these two in the identity

$$d(u, v) = \sum_{i,j} \frac{\partial(u, v)}{\partial(x_i, x_j)} d(x_i, x_j),$$

we find that $d(u, v)$ is a linear combination of the determinants of the matrix of ten† columns.

* This solution will not be a complete primitive unless a certain number of other arbitrary constants are involved as well as a, b, a supposition which is neither made nor excluded.

It may be well to point out that the solution here assumed consists of two equations, and not of one equation involving an arbitrary function; in fact, any solution whatever necessarily consists of two equations, and one point of the present method is that these are to be sought together, not successively.

† For n independent variables the number of columns in the matrix will be $\frac{1}{2}(n + 1)(n + 2)$, the number of rows being still three.

$$\left\| \begin{array}{cccccccc} d(x_1, x_2), & d(x_1, x_3), & . & . & . & d(x_i, x_j) & . & . & . & d(x_4, x_5) \\ A_{12}, & A_{13}, & . & . & . & A_{ij} & . & . & . & A_{45} \\ A'_{12}, & A'_{13}, & . & . & . & A'_{ij} & . & . & . & A'_{45} \end{array} \right\|$$

There are thus eight bidifferential expressions, and the problem is to be solved by finding such multiples of these as, when added together, will form a complete bidifferential.

§ 12. As in the case of LAGRANGE's linear equation, this will generally, in practice, be done by inspection, and the method will be useful for finding solutions in finite terms—when such exist. But in any case,* whether the inspection is successful or not, there can be no doubt of the existence of suitable multipliers, in infinite number. For it is certain that the equations (3) have—possibly among other solutions—an infinity of solutions, each involving two arbitrary constants at least, and any one of these may be written $u = a$, $v = b$, where a, b are the two constants; u, v are functions of the variables, but may, of course, be implicit functions of great complexity. The functions u, v must satisfy the conditions (4), and it immediately follows that $d(u, v)$ must be a linear combination of the determinants of the matrix formed from (4) as above; so that a corresponding system of multipliers must exist.

If the solution is not in finite terms it is not likely to be found by inspection, and it is quite probable that the best way to find it would be by solving the original equations (3) in series. By whatever means the solution is found, the corresponding system of multipliers is thereby determined.

If nine solutions of the form $u = a$, $v = b$ have been found, the nine bidifferentials $d(u_1, v_1)$, $d(u_2, v_2)$ . . . $d(u_9, v_9)$ must satisfy identically a linear relation, since they are all linear combinations of eight expressions only.

We shall say that one of the nine pairs of functions is a "bifunction" of the other eight pairs.

The following is, then, the definition of a bifunction. When the bidifferentials of any number of pairs of quantities are connected by an identical linear relation, with constant or variable coefficients, any one of these pairs is said to be a bifunction of the rest.

The word bifunction is simply used as an abbreviation—at least for the present. I am not without hope that at a future time it may be found to have some connotation.

* If one of the dependent variables with its derivatives is altogether absent from the equations (3), or if it can be made to disappear by a change of the other dependent variable, the equations (3) will in general have no solution. This case will then be excluded; it is the only case in which the method of solution in series (as given, for instance, by FRAU VON KOWALEVSKY, 'Crelle,' vol. 80) cannot be used to prove that solutions actually exist.

Another case that may fairly be excluded is that in which all the derivatives of one of the dependent variables do not occur or may be made to disappear by a change of the other. Such a system is equivalent to a single partial differential equation with one dependent variable, since the one whose derivatives are absent may be eliminated.

It is, of course, evident that if u, v are functions of variables $x_1, x_2 \ldots$ then the pair u, v is a bifunction of all the pairs that can be formed from $x_1, x_2 \ldots$ Other examples will be found later on in the paper.

§ 13. Sometimes solutions exist for systems of partial differential equations in which the number of dependent variables is less than the number of equations.

If, for instance, with the system just considered we take a third equation of the same form, the coefficients being distinguished by two dashes, there may be solutions common to the three equations. If $u = a$, $v = b$ give such a solution, then it follows in like manner that $d(u, v)$ is a linear combination of the determinants of the following matrix :—

$$\begin{Vmatrix} d(x_1, x_r) \ldots & d(x_i, x_j) \ldots \\ A_{12}, \ldots & A_{ij}, \ldots \\ A'_{12}, \ldots & A'_{ij}, \ldots \\ A''_{12}, \ldots & A''_{ij}, \ldots \end{Vmatrix}$$

Similarly for a greater number of equations.

Application to other Differential Equations.

§ 14. There are two classes of equations whose solution depends on that of a pair of such linear homogeneous equations as we have just been considering ; they are, firstly, systems of two equations in two dependent and two independent variables, and, secondly, equations of the second order with one dependent variable and two independent. We shall consider them in order.

Firstly, let y, z be the dependent variables and x_1, x_2 the independent ; sometimes we shall write x_3 for y and x_4 for z. Let p_1, p_2 be the partial derivatives of y and q_1, q_2 those of z, and let the equations be

$$f_1(x_1, x_2, y, z, p_1, p_2, q_1, q_2) = 0,$$

$$f_2(x_1, x_2, y, z, p_1, p_2, q_1, q_2) = 0.$$

A complete primitive will consist of two equations connecting x_1, x_2, y, z and involving four arbitrary constants. By differentiation these equations yield four more involving p_1, p_2, q_1, q_2. As the two equations are supposed to be a complete primitive it must be possible to find expressions for the four arbitrary constants in terms of x_1, x_2, y, z, p_1, q_1, p_2, q_2 ; the elimination of the four constants must give $f_1 = 0, f_2 = 0$.

Let a_1, a_2, a_3, a_4 be the constants, and u_1, u_2, u_3, u_4 the expressions for them in terms of x_1, x_2, y, z, p_1, q_1, p_2, q_2. Suppose f_3, f_4, f_5, f_6 to stand for u_1, u_2, u_3, u_4 respectively. Then by differentiation we have for any value of the suffix i from 1 to 6,

$$\frac{\partial f_i}{\partial x_r} + p_r\frac{\partial f_i}{\partial y} + q_r\frac{\partial f_i}{\partial z} + \frac{\partial f_i}{\partial p_1}\frac{dp_1}{dx_r} + \frac{\partial f_i}{\partial p_2}\frac{dp_2}{dx_r} + \frac{\partial f_i}{\partial q_1}\frac{dq_1}{dx_r} + \frac{\partial f_i}{\partial q_2}\frac{dq_2}{dx_r} = 0\ (r = 1, 2),$$

the letter d being used to denote differentiation with respect to x_1 or x_2 on the supposition that the other is constant, while ∂ indicates strictly partial differentiation.

Since $dp_2/dx_1 = dp_1/dx_2$, $dq_2/dx_1 = dq_1/dx_2$, we find by eliminating the derivatives of p_1, q_1, p_2, q_2, that

$$\begin{aligned}J(x_1, p_1, q_1, q_2) + p_1J(y, p_1, q_1, q_2) + q_1J(z, p_1, q_1, q_2) + J(x_2, p_2, q_1, q_2)\\ + p_2J(y, p_2, q_1, q_2) + q_2J(z, p_2, q_1, q_2) = 0,\end{aligned}$$

and
$$\begin{aligned}J(x_1, q_1, p_1, p_2) + p_1J(y, q_1, p_1, p_2) + q_1J(z, q_1, p_1, p_2) + J(x_2, q_2, p_1, p_2)\\ + p_2J(y, q_2, p_1, p_2) + q_2J(z, q_2, p_1, p_2) = 0\end{aligned}$$

where J( ) denotes the Jacobian of any four of the functions $f_1, f_2, f_3, f_4, f_5, f_6$ with respect to the variables specified in the bracket. Of these equations there are thirty, but since they are given by the elimination of six quantities from twelve equations only six of the thirty can be independent.

§ 15. One pair of these auxiliary equations will contain Jacobians of f_1, f_2, f_3, f_4, and will in fact express the conditions that the equations

$$dy = p_1dx_1 + p_2dx_2$$

$$dz = q_1dx_1 + q_2dx_2$$

shall be integrable without restriction when p_1, p_2, q_1, q_2 have the values given by the equations $f_1 = 0 = f_2$, $f_3 = a_1$, $f_4 = a_2$.

Thus, if a pair of functions f_3, f_4 can be found satisfying these two auxiliary equations, the solution can be completed by solving a pair of simultaneous ordinary equations. (See MAYER's method, FORSYTH, 'Theory of Differential Equations,' pp. 59–62.)

The two auxiliary equations that f_3, f_4 must satisfy are linear and homogeneous in their Jacobians, the coefficients of the Jacobians not involving the functions f_3, f_4; the number of independent variables is apparently eight, but it may be taken as six, since two of the eight variables x_1, x_2, y, z, p_1, p_2, q_1, q_2 are given as functions of the other six by the relations $f_1 = 0$, $f_2 = 0$, and may be supposed eliminated from f_3, f_4, if that is desirable.

The columns of the matrix formed as at § 11 are the rows of the following array :—

$$\begin{array}{llll} & d(x_1, x_2), & 0, & 0, \\ & d(x_1, y), & 0, & 0, \\ & d(x_1, z), & 0, & 0, \\ & d(x_2, y), & 0, & 0, \\ (5) & d(x_2, z), & 0, & 0, \end{array}$$

$$
\begin{array}{llll}
 & d(y, z), & 0, & 0, \\
 & d(x_1, p_1), & \{q_1, q_2\}, & \{p_2, q_1\}, \\
 & d(x_2, p_1), & 0, & \{p_2, q_2\}, \\
 & d(y, p_1), & p_1\{q_1, q_2\} & p_1\{p_2, q_1\} + p_2\{p_2, q_2\}, \\
(10) & d(z, p_1), & q_1\{q_1, q_2\}, & q_1\{p_2, q_1\} + q_2\{p_2, q_2\}, \\
 & d(x_1, p_2), & 0, & \{q_1, p_1\}, \\
 & d(x_2, p_2), & \{q_1, q_2\}, & \{q_2, p_1\}, \\
 & d(y, p_2), & p_2\{q_1, q_2\}, & p_1\{q_1, p_1\} + p_2\{q_2, p_1\}, \\
 & d(z, p_2), & q_2\{q_1, q_2\}, & q_1\{q_1, p_1\} + q_2\{q_2, p_1\}, \\
(15) & d(x_1, q_1), & \{q_2, p_1\}, & \{p_1, p_2\}, \\
 & d(x_2, q_1), & \{q_2, p_2\}, & 0, \\
 & d(y_1, q_1), & p_1\{q_2, p_1\} + p_2\{q_2, p_2\}, & p_1\{p_1, p_2\}, \\
 & d(z, q_1), & q_1\{q_2, p_1\} + q_2\{q_2, p_2\}, & q_1\{p_1, p_2\}, \\
 & d(x_1, q_2), & \{p_1, q_1\}, & 0, \\
(20) & d(x_2, q_2), & \{p_2, q_1\}, & \{p_1, p_2\}, \\
 & d(y, q_2), & p_1\{p_1, q_1\} + p_2\{p_2, q_1\}, & p_2\{p_1, p_2\}, \\
 & d(z, q_2), & q_1\{p_1, q_1\} + q_2\{p_2, q_1\}, & q_2\{p_1, p_2\}, \\
 & d(p_1, p_2), & 0, & \{x_1, q_1\} + p_1\{y, q_1\} + q_1\{z, q_1\} + \{x_2, q_2\} + p_2\{y, q_2\} + q_2\{z, q_2\}, \\
 & d(p_1, q_1), & \{x_1, q_2\} + p_1\{y, q_2\} + q_1\{z, q_2\}, & -\{x_1, p_2\} - p_1\{y, p_2\} - q_1\{z, p_2\}, \\
(25) & d(p_1, q_2), & -\{x_1, q_1\} - p_1\{y, q_1\} - q_1\{z, q_1\}, & -\{x_2, p_2\} - p_2\{y, p_2\} - q_2\{z, p_2\}, \\
 & d(p_2, q_1), & \{x_2, q_2\} + p_2\{y, q_2\} + q_2\{z, q_2\}, & \{x_1, p_1\} + p_1\{y, p_1\} + q_1\{z, p_1\}, \\
 & d(p_2, q_2), & -\{x_2, q_1\} - p_2\{y, q_1\} - q_2\{z, q_1\}, & \{x_2, p_1\} + p_2\{y, p_1\} + q_2\{z, p_1\}, \\
 & d(q_1, q_2), & \{x_1, p_1\} + p_1\{y, p_1\} + q_1\{z, p_1\} + \{x_2, p_2\} + p_2\{y, p_2\} + q_2\{z, q_2\}, & 0,
\end{array} \tag{5}
$$

Here $\{p_1, q_1\}$, for instance, is written for $\partial(f_1, f_2)/\partial(p_1, q_1)$, and every fifth row is numbered.

§ 16. In order, then, to solve the equations $f_1 = 0$, $f_2 = 0$ we have to form such a linear combination of the determinants of this array as will be a complete bidifferential, say $d(f_3, f_4)$, f_3, f_4 being such functions that the equations $f_1 = 0 = f_2$, $f_3 = a_3$, $f_4 = a_4$ can be solved for p_1, q_1, p_2, q_2. The array contains twenty-eight rows, but thirteen of these are combinations of the other fifteen. For instance, multiply the first row by $\partial f_1/\partial x_2$, the second by $\partial f_1/\partial y$, the third by $\partial f_1/\partial z$, the seventh by $\partial f_1/\partial p_1$, the eleventh by $\partial f_1/\partial p_2$, the fifteenth by $\partial f_1/\partial q_1$, the nineteenth by $\partial f_1/\partial q_2$ and add; the resulting row is

$$d(x_1, f_1),\ 0,\ 0,$$

which vanishes. Other vanishing rows may be formed similarly by combining the rows of the array so as to have in the first column one of the following—

$$d(x_1, f_1),\ d(x_2, f_1),\ d(y, f_1),\ d(z, f_1),\ d(p_1, f_1),\ d(p_2, f_1),\ d(q_1, f_1),\ d(q_2, f_1),$$
$$d(x_1, f_2),\ d(x_2, f_2),\ d(y, f_2),\ d(z, f_2),\ d(p_1, f_2),\ d(p_2, f_2),\ d(q_1, f_2),\ d(q_2, f_2).$$

The coefficients in these combinations are partial derivatives of f_1 or f_2, thus, for instance,

$$d(p_1, f_2) = \frac{\partial f_2}{\partial x_1} d(p_1, x_1) + \frac{\partial f_2}{\partial x_2} d(p_1, x_2) + \frac{\partial f_2}{\partial y} d(p_1, y) + \frac{\partial f_2}{\partial z} d(p_1, z)$$
$$+ \frac{\partial f_2}{\partial q_1} d(p_1, q_1) + \frac{\partial f_2}{\partial p_2} d(p_1, p_2) + \frac{\partial f_2}{\partial q_2} d(p_1, q_2),$$

and so in other cases.

The number of these combinations is sixteen, but it is to be lowered by three, since $d(f_1, f_1)$ and $d(f_2, f_2)$ are identically zero and $d(f_1, f_2)$ can be formed by combining the sixteen in two ways, so that three linear combinations of the sixteen bidifferentials vanish identically.

Hence the array contains virtually only fifteen rows $(28 - 13)$ and as there are three columns, we have thirteen bidifferential expressions to combine. Any pair of the four functions x_1, x_2, y, z will satisfy the two auxiliary equations, as is clear either from the equations themselves or from an examination of the matrix; of course these solutions of the auxiliary equations will not give a complete primitive.

§ 17. If a complete primitive has been found it leads, as has been explained, to four equations

$$u_1 = a_1, \; u_2 = a_2, \; u_3 = a_3, \; u_4 = a_4,$$

and any pair of these must satisfy the auxiliary equations. Thus twelve pairs of functions satisfying these are known, namely

$$x_i \text{ and } x_j \; (i, j = 1, 2, 3, 4)$$
$$u_i \text{ and } u_j \; (i, j = 1, 2, 3, 4).$$

These, however, are not all independent, but one pair is a bifunction of the other eleven.

For if

$$\left.\begin{aligned} \phi(x_1, x_2, x_3, x_4, a_1, a_2, a_3, a_4) &= 0 \\ \psi(x_1, x_2, x_3, x_4, a_1, a_2, a_3, a_4) &= 0 \end{aligned}\right\} \quad \ldots \ldots \ldots \quad (6)$$

are the equations of the complete primitive, they must reduce to identities when u_1, u_2, u_3, u_4 are substituted for a_1, a_2, a_3, a_4 respectively.

Hence

$$\left.\begin{aligned} \phi(x_1, x_2, x_3, x_4, u_1, u_2, u_3, u_4) &= 0 \\ \psi(x_1, x_2, x_3, x_4, u_1, u_2, u_3, u_4) &= 0 \end{aligned}\right\} \quad \ldots \ldots \ldots \quad (7),$$

identically, and

$$\sum_i \frac{\partial \phi}{\partial x_i} dx_i = - \sum_i \frac{\partial \phi}{\partial u_i} du_i,$$

$$\sum_i \frac{\partial \psi}{\partial x_i} dx_i = - \sum_i \frac{\partial \psi}{\partial u_i} du_i,$$

so that

$$\sum_{i, j} \frac{\partial(\phi, \psi)}{\partial(x_i, x_j)} d(x_i, x_j) = \sum_{i, j} \frac{\partial(\phi, \psi)}{\partial(u_i, u_j)} d(u_i, u_j) \quad \ldots \ldots \quad (8),$$

and the bidifferentials of the twelve pairs of functions are connected by a linear relation.

§ 18. The method of CHARPIT for a single partial differential equation of the first order shows how all solutions may be deduced from one complete primitive, and it is a question of interest and importance whether there is any analogous method for simultaneous equations. Now it follows at once from the conditions for a complete bidifferential that a bifunction of the pairs that can be formed from m functions, say $u_1, u_2 \ldots\ldots u_m$, will be a pair of functions of $u_1 \ldots\ldots u_m$. In the present case a bifunction of the six pairs that can be formed with u_1, u_2, u_3, u_4 will be a pair of functions of these four, and the complete primitive to which it will lead will be the same as that given by u_1, u_2. For when a solution of the auxiliary equations is known it leads directly to one and only one complete primitive by the integration of the equations*

$$dy = p_1\,dx_1 + p_2\,dx_2$$
$$dz = q_1\,dx_1 + q_2\,dx_2\,;$$

also the complete primitive to which the equations $F_1(u_1, u_2, u_3, u_4) = \text{const.}$, $F_2(u_1, u_2, u_3, u_4) = \text{const.}$, will lead can be no other than is given by

$$u_1 = a_1, \quad u_2 = a_2, \quad u_3 = a_3, \quad u_4 = a_4.$$

It must not, however, be forgotten that the system $F_1 = \text{const.}$, $F_2 = \text{const.}$, $f_1 = 0$, $f_2 = 0$ may have a singular solution. If F_1, F_2 involve two other arbitrary constants this singular solution will involve four, and therefore in general be a complete primitive of the equations $f_1 = 0, f_2 = 0$. Moreover, all new complete primitives are included among those thus given.

For every solution implies six equations connecting $x_1, x_2, y, z, p_1, q_1, p_2, q_2$ (two of these six are of course $f_1 = 0$, $f_2 = 0$), and, therefore, by elimination of $x_1, x_2, y, z, p_1, q_1, p_2, q_2$, two equations or more connecting u_1, u_2, u_3, u_4, which are known in terms of these eight quantities. If u_1, u_2, u_3, u_4 are connected by four equations they are constants, and the solution is therefore included in the old complete primitive. Let us, then, suppose that u_1, u_2, u_3, u_4 are connected by two or by three equations,

$$F_\alpha(u_1, u_2, u_3, u_4) = 0 \qquad (\alpha = 1, 2 \text{ or } 1, 2, 3).$$

Now if p_1, p_2, q_1, q_2, are all expressed in terms of p, q, two of their number, and x_1, x_2, y, z, by means of the equations $f_1 = 0, f_2 = 0$, the expressions

$$dy - p_1dx_1 - p_2dx_2,\ dz - q_1dx_1 - q_2dx_2$$

must both be expressible in the form

$$A_1du_1 + A_2du_2 + A_3du_3 + A_4du_4,$$

* Otherwise thus—if in the auxiliary equations we suppose f_3 to have the known value u_1, they become a pair of linear equations for f_4, which must be satisfied by u_2, u_3, u_4; now two linear equations in six independent variables can only have four functionally independent solutions, and one of these is known, namely, u_1. (In exceptional cases the two linear equations for u_2, u_3, u_4 may be equivalent; for instance, suppose $f_1 = p_1 + q_1$, $u_1 = p_2 + q_2$, f_2 having any form.) Hence, except in special cases, the particular complete primitive is defined when one of the functions u_1, u_2, u_3, u_4, or more generally a combination of them, $F(u_1, u_2, u_3, u_4)$ is known. In the case supposed in the text two such combinations are known.

and since dp, dq are absent we must have in each case

$$\sum_{r=1}^{4} \mathrm{A}_r \frac{\partial u_r}{\partial p} = 0, \quad \sum_{r=1}^{4} \mathrm{A}_r \frac{\partial u_r}{\partial q} = 0.$$

Thus the equations

$$dy = p_1 dx_1 + p_2 dx_2, \quad dz = q_1 dx_1 + q_2 dx_2$$

become

$$\left\| \begin{matrix} du_1, & du_2, & du_3, & du_4, \\ \frac{\partial u_1}{\partial p}, & \frac{\partial u_2}{\partial p}, & \frac{\partial u_3}{\partial p}, & \frac{\partial u_4}{\partial p} \\ \frac{\partial u_1}{\partial q}, & \frac{\partial u_2}{\partial q}, & \frac{\partial u_3}{\partial q}, & \frac{\partial u_4}{\partial q} \end{matrix} \right\| = 0.$$

These two equations, connecting du_1, du_2, du_3, du_4, taken with the system

$$\sum_{r=1}^{4} \frac{\partial \mathrm{F}_\alpha}{\partial u_r} du_r = 0 \ (\alpha = 1, 2 \text{ or } 1, 2, 3),$$

show that if u_1, u_2, u_3, u_4 satisfy by themselves no other relations than $\mathrm{F}_\alpha = 0$ ($\alpha = 1$, 2 or 1, 2, 3) we must have, as a consequence of the equations of the solution,

$$\sum_{r=1}^{4} \frac{\partial \mathrm{F}_\alpha}{\partial u_r} \frac{\partial u}{\partial p} = 0, \quad \sum_{r=1}^{4} \frac{\partial \mathrm{F}_\alpha}{\partial u_r} \frac{\partial u_r}{\partial q} = 0.$$

If, then, there are two equations

$$\mathrm{F}_1 = 0, \quad \mathrm{F}_2 = 0,$$

the four equations

$$\sum_{r=1}^{4} \frac{\partial \mathrm{F}_\alpha}{\partial u_r} \frac{\partial u_r}{\partial p} = 0, \quad \sum_{r=1}^{4} \frac{\partial \mathrm{F}_\alpha}{\partial u_r} \frac{\partial u_r}{\partial q} = 0 \ (\alpha = 1, 2)$$

must reduce to two only. This will be the ordinary case, and we see that if the forms of F_1, F_2, have been found by any means, the solution is completed without integration; the process corresponds to CHARPIT's method of deducing all complete primitives from one, but it differs in that the functions F_1, F_2, are not arbitrary; they must, in fact, be so chosen that the four equations last written shall reduce to two, and the conditions for this are clearly very complicated in general, though in particular cases available forms for F_1, F_2 may be seen on inspection.

In the more uncommon case, when there are three equations

$$\mathrm{F}_1 = 0, \ \mathrm{F}_2 = 0, \ \mathrm{F}_3 = 0,$$

the six equations

$$\sum_{r=1}^{4} \frac{\partial \mathrm{F}_\alpha}{\partial u_r} \frac{\partial u_r}{\partial p} = 0, \quad \sum_{r=1}^{4} \frac{\partial \mathrm{F}_\alpha}{\partial u_r} \frac{\partial u_r}{\partial q} = 0 \ (\alpha = 1, 2, 3),$$

must reduce to one only.

These two cases are further discussed, from a somewhat different point of view, in §§ 21—23.

It should not be forgotten that the form in which the new complete primitive has

just appeared is not that in which complete primitives were discussed in § 14, since the equations are not here supposed to be solved for the arbitrary constants.

§ 19. In addition to the six pairs (u_i, u_j) of functions satisfying the auxiliary equations, we have also the six pairs (x_i, x_j); of these twelve, eleven are independent, the other being a bifunction of them. If we can find a bifunction of the eleven pairs which is not a bifunction of either set of six it will give a new complete primitive; whether every, or indeed any, other primitive is thus given is a matter for further inquiry.

Suppose $v_i = b_i$ $(i = 1, 2, 3, 4)$ to be a new complete primitive, then it gives six more pairs of functions satisfying the auxiliary equations, and thus we have in all eighteen pairs. The bidifferentials of these must be connected by $(18-13)$ five linear relations, one of which has been written (8); by means of the other four, an expression of either of the following forms—

$$\mathrm{A}d(v_1, v_2) + \mathrm{B}d(v_1, v_3) + \mathrm{C}d(v_1, v_4),$$
$$\mathrm{A}d(v_2, v_3) + \mathrm{B}d(v_3, v_1) + \mathrm{C}d(v_1, v_2),$$

can be found which will be equal to a linear combination of the twelve bidifferentials $d\,(x_i, x_j)$ and $d\,(u_i, u_j)$. It is natural to ask whether, conversely, any linear combination of these twelve which can be written in one of the above forms will lead to a complete primitive? In the first case this is not so, for if we take any function whatever, η, of six independent variables, $\xi_1, \xi_2 \ldots \xi_6$, we may choose the coefficients $\alpha_1, \ldots \alpha_6$, so that

$$\sum_{i=1}^{6} \alpha_i\, d(\eta, \xi_i)$$

shall be a linear combination of eleven* given bidifferentials; the expression $\Sigma\, \alpha_i\, d\xi_i$ may then be reduced to three terms, $\beta_1\, d\zeta_1 + \beta_2\, d\zeta_2 + \beta_3\, d\zeta_3$, so that for an *arbitrary* function (η) a combination of the eleven given bidifferentials can be found of the form $\beta_1\, d(\eta, \zeta_1) + \beta_2\, d(\eta, \zeta_2) + \beta_3\, d(\eta, \zeta_3)$, which is the same as $\mathrm{A}d(v_1, v_2) + \mathrm{B}d(v_1, v_3) + \mathrm{C}d(v_1, v_4)$. This argument does not apply to the second form

$$\mathrm{A}d(v_2, v_3) + \mathrm{B}d(v_3, v_1) + \mathrm{C}d(v_1, v_2),$$

and further investigation may show that any combination of the eleven that can be reduced to this form† will lead to a primitive.

* Not of any lower number in general, since the most general bidifferential expression in this number of variables contains fifteen terms, while the expression just written vanishes identically if

$$\alpha_i = \partial\eta/\partial\xi_i,$$

so that there are virtually only five coefficients, of which one must be left arbitrary.

† The conditions necessary that a bidifferential expression may be reducible to this form include algebraic ones which are the same as for a complete bidifferential, since

$$\mathrm{A}d(v_2, v_3) + \mathrm{B}d(v_3, v_1) + \mathrm{C}d(v_1, v_2) = \frac{1}{\mathrm{A}}\left\{\mathrm{A}dv_2 - \mathrm{B}dv_1\right\}\left\{\mathrm{A}dv_3 - \mathrm{C}dv_1\right\}.$$

§ 20. Before we can claim in any sense to have found the general solution of the auxiliary equations, we must be in possession of thirteen pairs of functions satisfying them; we have only eleven when we know one complete primitive, and hence one more complete primitive, or even possibly two, must be found. An example (below, § 29) will show that one more is not always enough.

It is perhaps worth while to remark that any complete primitive defines the whole system of solutions, since it defines the differential equations.

§ 21. The question of finding new solutions when a complete primitive is known may be attacked by the method of varying the parameters. Take the equations (6) or (7) of § 17. The problem is then to find such variable values for u_1, u_2, u_3, u_4 as will satisfy the equations

$$\sum_{i=1}^{4} \frac{\partial \phi}{\partial u_i} du_i = 0, \ \sum_{i=1}^{4} \frac{\partial \psi}{\partial u_i} du_i = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9).$$

Since all variables are supposed functions of x_1, x_2, we may make one of two suppositions with respect to u_1, u_2, u_3, u_4; either they are connected by three relations and are all functions of the same variable, say t, which is of course a function of x_1, x_2, or they are only connected by two relations, so that two of them may be taken as functions of the other two.

Suppose first that they are all functions of the one variable t. Then, generally, the four equations (7), (9) will define x_1, x_2, x_3, x_4 also as functions of t, and hence this supposition is not admissible unless it is possible to choose the functions of t in such a way* that the four equations (7), (9) will be only equivalent to three. The

If these conditions are satisfied by an expression

$$\sum_{i, j = 1, 2 \,.\,.\, 6} A_{ij} d(x_i, x_j),$$

it can be put in the form

$$\left(\sum_{i=1}^{6} \lambda_i \, dx_i\right)\left(\sum_{i=1}^{6} \mu_i \, dx_i\right),$$

and then it must further be possible to express

$$\sum_{i=1}^{6} \lambda_i \, dx_i \text{ and } \sum_{i=1}^{6} \mu_i \, dx_i$$

as linear combinations of *three* differentials, dv_1, dv_2, dv_3. The discussion of the conditions therefore belongs to the theory of the reduction of two such expressions, that is, of the extended PFAFF problem.

* It seems obvious that this will not generally be possible; but it may be well to give an example. Suppose the complete primitive to be

$$\left.\begin{aligned} y &= ax_1^2 + bx_2 + c, \\ z &= cx_1 + ex_2^2 + bx_1x_2^2 \end{aligned}\right\},$$

so that the differential equations are

$$\begin{aligned} y &= \tfrac{1}{2}p_1x_1 + p_2x_2 + q_1 - p_2x_2^2, \\ z &= q_1x_1 + \tfrac{1}{2}q_2x_2 - p_2x_1x_2^2, \end{aligned}$$

then the variations of the parameters a, b, c, e must satisfy the equations,

$$\begin{aligned} x_1^2da + x_2db + dc &= 0 \\ x_1dc + x_2^2de + x_1x_2^2db &= 0; \end{aligned}$$

number of conditions, which will be of the nature of ordinary differential equations, thus imposed on the four parameters must not be greater than three ; for if they are subjected to four conditions they are made invariable ; it may be, however, less than three. For instance, a complete primitive of the equations $p_1 = p_2$, $q_1 = q_2$ is given by

$$y = a(x_1 + x_2) + b, \ z = c(x_1 + x_2) + e\,;$$

the equations given by varying the parameters are

$$(x_1 + x_2)da + db = 0,$$
$$(x_1 + x_2)dc + de = 0,$$

which give the *single* differential equation connecting the parameters

$$da\, de = db\, dc.$$

We may then assume arbitrary forms for two parameters in terms of a third, and find the fourth by integration. Say, for instance,

$$b = \phi(a), \ c = \psi(a),$$

then

$$e = \int \phi'(a)\psi'(a)da,$$
$$x_1 + x_2 = -\phi'(a)\,;$$

thus we arrive at the known general solution

$$y = \chi(x_1 + x_2), \ z = \omega(x_1 + x_2).$$

---

whence, by elimination of x_1,

$$(x_2^2 db + dc)^2(x_2 db + dc) + x_2^4 de^2 da = 0.$$

This equation must fail to define x_2, so that b, c, and a or e must be constant ; thence it follows that all four parameters must be constant.

I lay stress on this, because it is not in agreement with the results of Professor KÖNIGSBERGER ('Crelle,' vol. 109, p. 318), and appears in fact to show that his method there given is faulty. Professor KÖNIGSBERGER assumes (p. 313 ; I take $m = 2$) that the most general integral of the equations

$$f_1(x_1, x_2, y, z, p_1, p_2, q_1, q_2) = 0$$
$$f_2(x_1, x_2, y, z, p_1, p_2, q_1, q_2) = 0$$

has the form

$$y = \omega_1(x_1, x_2, \phi_1[\psi_1(x_1, x_2)], \phi_2[\psi_2(x_1, x_2)])$$
$$z = \omega_2(x_1, x_2, \phi_1[\psi_1(x_1, x_2)], \phi_2[\psi_2(x_1, x_2)]),$$

where ϕ_1, ϕ_2 denote arbitrary and ψ_1, ψ_2 definite functions. But suppose these equations solved for ϕ_1, ϕ_2 in the form

$$\phi_1[\psi_1(x_1, x_2)] = \chi_1(x_1, x_2, y, z)$$
$$\phi_2[\psi_2(x_1, x_2)] = \chi_2(x_1, x_2, y, z)$$

and the arbitrary functions eliminated by differentiation. The differential equations thus formed are of the first degree in p_1, p_2, q_1, q_2, and are not by any means of the general form assumed. The differential equations in the examples given by Professor KÖNIGSBERGER are, in fact, linear (see pp. 319, 328). The method appears to be founded on an interpretation of the last clause of § 2 (p. 290), which is not justified.

In the case of two equations of CLAIRAUT's form

$$y = p_1 x_1 + p_2 x_2 + \phi(p_1, p_2, q_1, q_2),$$
$$z = q_1 x_1 + q_2 x_2 + \psi(p_1, p_2, q_1, q_2),$$

which will be more fully considered later, the number of differential relations among the parameters is *two*, so that one parameter may be taken as an arbitrary function of a second, and the other two found in terms of the second by solving two ordinary differential equations.

If the primitive* is

$$y = a\alpha + b\beta + c\gamma + e\delta$$
$$z = \mathrm{A}\alpha + \mathrm{B}\beta + \mathrm{C}\gamma + \mathrm{E}\delta,$$

where a, b, c, e are the parameters, A, B, C, E known functions of a, b, c, e, and α, β, γ, δ known functions of x_1, x_2, then the variations of the parameters must satisfy the relations

$$\alpha da + \beta db + \gamma dc + \delta de = 0,$$
$$\alpha d\mathrm{A} + \beta d\mathrm{B} + \gamma d\mathrm{C} + \delta d\mathrm{E} = 0,$$

and thus, in general, if a, b, c, e are all functions of one variable they are connected by *three* relations

$$d\mathrm{A}/da = d\mathrm{B}/db = d\mathrm{C}/dc = d\mathrm{E}/de.$$

The integral equivalent of these equations consists of three relations connecting a, b, c, e with three arbitrary constants, and by eliminating a, b, c, e we find a new solution of the original differential equations which is not a complete primitive, since it only contains three arbitrary constants.

These examples show that the number of conditions to be fulfilled by the parameters when all four are taken to be functions of one of them, may be one, two, or three; this number is to be made up to three by assuming arbitrary relations (two, one, or none, as the case may be).

§ 22. Usually the parameters will not be functions of one variable only, and we may suppose two of them, u_3, u_4, to be functions of the other two, u_1, u_2.

The partial differential coefficients

$$\frac{du_3}{du_1}, \frac{du_3}{du_2}, \frac{du_4}{du_1}, \frac{du_4}{du_2}$$

are then given by the equations (9), each of which is equivalent to two. The first, for instance, gives

$$\frac{\partial\phi}{\partial u_1} + \frac{\partial\phi}{\partial u_3}\frac{du_3}{du_1} + \frac{\partial\phi}{\partial u_4}\frac{du_4}{du_1} = 0,$$
$$\frac{\partial\phi}{\partial u_2} + \frac{\partial\phi}{\partial u_3}\frac{du_3}{du_2} + \frac{\partial\phi}{\partial u_4}\frac{du_4}{du_2} = 0.$$

The derivatives are thus given in terms of u_1, u_2, u_3, u_4, x_1, x_2, x_3, x_4, and the last

* It is unnecessary to give the differential equations.

four may be eliminated by means of the relations (7); so that in the end we shall have two relations connecting u_1, u_2, u_3, u_4, and the derivatives; the problem is of the same form as the original one, to solve two simultaneous partial differential equations in two dependent and two independent variables.

Interchange of Variables and Parameters.

§ 23. A curious thing may be noticed at this point. If in the equations $\phi = 0$, $\psi = 0$, we treat x_1, x_2, x_3, x_4 as arbitrary constants and eliminate them by differentiation, we are led to the same differential equations connecting u_1, u_2, u_3, u_4 as were just now given by the variation of parameters. Thus two equations in two sets of four quantities will give two pairs of simultaneous partial differential equations by taking each set of the quantities in turn as variables and the other as arbitrary constants. The auxiliary equations, if expressed in terms of the eight quantities, will be the same in both cases; this gives a meaning to the six solutions of the form (x_i, x_j) which we found the auxiliary equations to have, for any one of the six will lead to the primitive $\phi = 0$, $\psi = 0$ of the second pair of differential equations, just as a solution (u_i, u_j) leads to this primitive for the first pair; any new solution of the auxiliary equations will in general lead to a new complete primitive for either pair, but an exception to this rule will arise when, for instance, the x differential equations have a complete primitive which gives three relations among u_1, u_2, u_3, u_4.

The array (5), transformed so that the variables are x_1, x_2, x_3, x_4, u_1, u_2, u_3, u_4, connected by the equations $\phi = 0$, $\psi = 0$, will have six rows of the form

$$d(x_i, x_j),\ 0,\ 0,$$

six of the form

$$d(u_i, u_j),\ 0,\ 0,$$

and in the other sixteen there will be

$$d(x_i, u_j) \text{ in the first column,}$$

in the second the minor of $\dfrac{\partial^2\phi}{\partial x_i \partial u_j}$ in the determinant:

$$\begin{vmatrix} \dfrac{\partial^2\phi}{\partial x_1\partial u_1}, & \dfrac{\partial^2\phi}{\partial x_1\partial u_2}, & \dfrac{\partial^2\phi}{\partial x_1\partial u_3}, & \dfrac{\partial^2\phi}{\partial x_1\partial u_4}, & \dfrac{\partial\phi}{\partial x_1}, & \dfrac{\partial\psi}{\partial x_1} \\ \dfrac{\partial^2\phi}{\partial x_2\partial u_1}, & \dfrac{\partial^2\phi}{\partial x_2\partial u_2}, & \dfrac{\partial^2\phi}{\partial x_2\partial u_3}, & \dfrac{\partial^2\phi}{\partial x_2\partial u_4}, & \dfrac{\partial\phi}{\partial x_2}, & \dfrac{\partial\psi}{\partial x_2} \\ \dfrac{\partial^2\phi}{\partial x_3\partial u_1}, & \dfrac{\partial^2\phi}{\partial x_3\partial u_2}, & \dfrac{\partial^2\phi}{\partial x_3\partial u_3}, & \dfrac{\partial^2\phi}{\partial x_3\partial u_4}, & \dfrac{\partial\phi}{\partial x_3}, & \dfrac{\partial\psi}{\partial x_3} \\ \dfrac{\partial^2\phi}{\partial x_4\partial u_1}, & \dfrac{\partial^2\phi}{\partial x_4\partial u_2}, & \dfrac{\partial^2\phi}{\partial x_4\partial u_3}, & \dfrac{\partial^2\phi}{\partial x_4\partial u_4}, & \dfrac{\partial\phi}{\partial x_4}, & \dfrac{\partial\psi}{\partial x_4} \\ \dfrac{\partial\phi}{\partial u_1}, & \dfrac{\partial\phi}{\partial u_2}, & \dfrac{\partial\phi}{\partial u_3}, & \dfrac{\partial\phi}{\partial u_4}, & 0, & 0 \\ \dfrac{\partial\psi}{\partial u_1}, & \dfrac{\partial\psi}{\partial u_2}, & \dfrac{\partial\psi}{\partial u_3}, & \dfrac{\partial\psi}{\partial u_4}, & 0, & 0 \end{vmatrix} \quad \ldots\ldots \quad (10)$$

in the third the same expression with ϕ, ψ interchanged. The array is thus practically unchanged by interchanging the sets x and u, as should be the case.

§ 24. This transformation may be accomplished by taking the equations

$$\sum_i \frac{\partial\phi}{\partial u_i}\frac{du_i}{dx_1} = 0 = \sum_i \frac{\partial\phi}{\partial u_i}\frac{du_i}{dx_2},$$

from which may be deduced

$$\frac{d}{dx_1}\left(\sum_i \frac{\partial\phi}{\partial u_i}\frac{du_i}{dx_2}\right) = \frac{d}{dx_2}\left(\sum_i \frac{\partial\phi}{\partial u_i}\frac{du_i}{dx_1}\right),$$

or

$$\sum_i\sum_j \frac{\partial^2\phi}{\partial u_i\partial u_j}\frac{du_i}{dx_2}\frac{du_j}{dx_1} + \sum_i \frac{\partial^2\phi}{\partial u_i\partial x_1}\frac{du_i}{dx_2} + \sum_i \frac{\partial^2\phi}{\partial u_i\partial y} p_1 \frac{du_i}{dx_2} + \sum_i \frac{\partial^2\phi}{\partial u_i\partial z} q_1 \frac{du_i}{dx_2}$$

$$= \sum_i\sum_j \frac{\partial^2\phi}{\partial u_i\partial u_j}\frac{du_i}{dx_1}\frac{du_j}{dx_2} + \sum_i \frac{\partial^2\phi}{\partial u_i\partial x_2}\frac{du_i}{dx_1} + \sum_i \frac{\partial^2\phi}{\partial u_i\partial y} p_2 \frac{du_i}{dx_1} + \sum_i \frac{\partial^2\phi}{\partial u_i\partial z} q_2 \frac{du_i}{dx_1}.$$

Now p_1, q_1, p_2, q_2 are given by the relations

$$\frac{\partial\phi}{\partial x_1} + p_1\frac{\partial\phi}{\partial y} + q_1\frac{\partial\phi}{\partial z} = 0, \text{ \&c.},$$

and hence this equation may be written

$$\sum_i \left\{\frac{du_i}{dx_2}\,\frac{\partial\left(\phi, \psi, \frac{\partial\phi}{\partial u_i}\right)}{\partial(x_1, y, z)}\right\} = \sum_i \left\{\frac{dui}{dx_1}\,\frac{\partial\left(\phi, \psi, \frac{\partial\phi}{du_i}\right)}{d(x_2, y, z)}\right\} \quad . \quad . \quad . \quad . \quad . \quad (11);$$

in this ϕ, ψ may be interchanged so as to give another equation.

Now, suppose $\theta = a_1$, $\chi = a_2$ to be two of the four equations connecting u_1, u_2, u_3, u_4 with x_1, x_2, which yield a new complete primitive, and that y, z have been eliminated from θ, χ by means of the equations $\phi = 0$, $\psi = 0$, then the derivatives $\frac{du_1}{dx_1}$, $\frac{du_2}{dx_1}$, &c., are given by the following relations:—

$$\sum_i \frac{\partial\phi}{\partial u_i}\frac{du_i}{dx_1} = 0,$$

$$\sum_i \frac{\partial\psi}{\partial u_i}\frac{du_i}{dx_1} = 0,$$

$$\sum_i \frac{\partial\theta}{\partial u_i}\frac{du_i}{dx_1} + \frac{\partial\theta}{\partial x_1} = 0,$$

$$\sum_i \frac{\partial\chi}{\partial u_i}\frac{du_i}{dx_1} + \frac{\partial\chi}{\partial x_1} = 0,$$

and similarly for the derivatives with respect to x_2.

Substituting the values hence found for these derivatives in the equation (11), we have an equation linear in the Jacobians of the form

$$\frac{\partial(\theta, \chi)}{\partial(x_i, u_j)}\ (i = 1, 2;\ j = 1, 2, 3, 4),$$

the coefficient of the Jacobian written being the minor of $\frac{\partial^2\phi}{\partial x_i \partial u_j}$ in the determinant (10). Hence the constituents in the second column of the transformed array are as stated, and those of the third are found in like manner. It is not, of course, necessary that these columns should be the same as would be found by actual substitution of the values of p_1, p_2, q_1, q_2 in the columns of the original array; a linear transformation is allowable, with constant or variable coefficients.

The above process gives fifteen independent rows of the array; the others are deduced from the consideration that y, z are known in terms of $x_1, x_2, u_1, u_2, u_3, u_4$ from the equations $\phi = 0$, $\psi = 0$.

Examples.

§ 25. I. As a first example of the method of solution, take the equations

$$\alpha_1 = \alpha_2, \; \beta_1 = \beta_2,$$

where α_1, β_1 denote known functions of x_1, p_1, q_1 and α_2, β_2 known functions of x_2, p_2, q_2.

In the array (5) multiply the seventh row by $\frac{\partial(\alpha_1, \beta_1)}{\partial(x_1, p_1)}$, the fifteenth by $\frac{\partial(\alpha_1, \beta_1)}{\partial(x_1, q_1)}$, the twenty-fourth by $\frac{\partial(\alpha_1, \beta_1)}{\partial(p_1, q_1)}$, and add. The result in the first column is $d(\alpha_1, \beta_1)$, in the second, by virtue of the particular forms of f_1 and f_2,

$$\{x_1, p_1\}\{q_1, q_2\} + \{x_1, q_1\}\{q_2, p_1\} + \{p_1, q_1\}\{x_1, q_2\} \text{ or } 0,$$

and in the third,

$$\{x_1, p_1\}\{p_2, q_1\} + \{x_1, q_1\}\{p_1, p_2\} - \{p_1, q_1\}\{x_1, p_2\} \text{ or } 0.$$

Hence α_1, β_1 are two functions satisfying the auxiliary equations, and a solution is given by finding p_1, q_1, p_2, q_2 from the equations

$$\alpha_1 = \alpha_2 = a,$$
$$\beta_1 = \beta_2 = b,$$

and integrating. Two constants will be introduced by integration, so that the result is a complete primitive.

§ 26. II. Take, secondly, the equations

$$y = p_1 x_1 + \mathrm{F}(x_2, p_1, q_1, p_2, q_2),$$
$$z = q_1 x_1 + \mathrm{G}(x_2, p_1, q_1, p_2, q_2).$$

Here the twenty-fourth row is

$$d(p_1, q_1), \; 0, \; 0,$$

so that p_1, q_1 are two functions satisfying the auxiliary equations, and the integral is to be found by putting $p_1 = a$, $q_1 = b$. Thus we have

$$y - ax_1 = \mathrm{F}(x_2, a, b, p_2, q_2),$$
$$z - bx_1 = \mathrm{G}(x_2, a, b, p_2, q_2),$$

or

$$\eta = \mathrm{F}(\xi, a, b, \eta', \zeta'),$$
$$\zeta = \mathrm{G}(\xi, a, b, \eta', \zeta'),$$

where

$$\xi = x_2,\ \eta = y - ax_1,\ \zeta = z - bx_1,$$
$$\eta' = d\eta/d\xi,\ \zeta' = d\zeta/d\xi.$$

These are ordinary differential equations, the solution of which will involve two new arbitrary constants and so constitute a complete primitive of the original equations.

§ 27. III. The equations

$$y = p_1x_1 + p_2x_2 + \phi(p_1, p_2, q_1, q_2),$$
$$z = q_1x_1 + q_2x_2 + \psi(p_1, p_2, q_1, q_2),$$

are of special interest, because more complete primitives than one can be found. The obvious solution is $p_1 = a_1$, $p_2 = a_2$, $q_1 = b_1$, $q_2 = b_2$,

$$y = a_1x_1 + a_2x_2 + \phi(a_1, a_2, b_1, b_2),$$
$$z = b_1x_1 + b_2x_2 + \psi(a_1, a_2, b_1, b_2).$$

Suppose a_1, a_2, b_1, b_2 to be variable, but functions of one variable only—say a_1, then their variations must satisfy the relations

$$x_1\,da_1 + x_2\,da_2 + d\phi = 0,$$
$$x_1\,db_1 + x_2\,db_2 + d\psi = 0.$$

These define x_1, x_2, and, therefore, also y, z as functions of a_1, unless the determinants of the matrix

$$\left\| \begin{matrix} da_1, da_2, d\phi \\ db_1, db_2, d\psi \end{matrix} \right\|$$

vanish; it is necessary, then, that these determinants should vanish. Thus a_1, b_1, a_2, b_2 are connected by two ordinary differential equations. We may assume any third relation connecting them at will; suppose $b_1 = \mathrm{F}(a_1)$, F denoting an arbitrary function. Then by integration we may suppose a_2, b_2 found in terms of a_1.

Also a_1 is connected with x_1, x_2 by the relation

$$x_1 + x_2\frac{da_2}{da_1} + \frac{d\phi}{da_1} = 0,$$

so that a_1, b_1, a_2, b_2 are all known in terms of x_1, x_2, and by substitution the values of y, z are found.

§ 28. The solution may be verified. We have taken $b_1 = F(a_1)$, a known but arbitrary function of a_1, and a_2, b_2 other functions of a_1, such that

$$\frac{db_2}{da_1} = F'(a_1)\frac{da_2}{da_1}$$

$$\left[\frac{\partial\psi}{\partial a_1} + \frac{\partial\psi}{\partial b_1}F'(a_1) + \frac{\partial\psi}{\partial a_2}\frac{da_2}{da_1} + \frac{\partial\psi}{\partial b_2}\frac{db_2}{da_1}\right] = F'(a_1)\left[\frac{\partial\phi}{\partial a_1} + \frac{\partial\phi}{\partial b_1}F'(a_1) + \frac{\partial\phi}{\partial a_2}\frac{da_2}{da_1} + \frac{\partial\phi}{\partial b_2}\frac{db_2}{da_1}\right].$$

Then we have the further relations

$$x_1 + x_2\frac{da_2}{da_1} + \frac{\partial\phi}{\partial a_1} + \frac{\partial\phi}{db_1}F'(a_1) + \frac{\partial\phi}{\partial a_2}\frac{da_2}{da_1} + \frac{\partial\phi}{\partial b_2}\frac{db_2}{da_1} = 0,$$

$$x_1 F'(a_1) + x_2\frac{db_2}{da_1} + \frac{\partial\psi}{\partial a_1} + \frac{\partial\psi}{\partial b_1}F'(a_1) + \frac{\partial\psi}{\partial a_2}\frac{da_2}{da_1} + \frac{\partial\psi}{db_2}\frac{db_2}{da_1} = 0,$$

which are of course not distinct. Also

$$y = a_1 x_1 + a_2 x_2 + \phi(a_1, a_2, b_1, b_2),$$

so that

$$p_1 = a_1 + \frac{da_1}{dx_1}\left[x_1 + x_2\frac{da_2}{da_1} + \frac{\partial\phi}{\partial a_1} + \frac{\partial\phi}{\partial a_2}\frac{da_2}{da_1} + \frac{\partial\phi}{\partial b_1}\frac{db_1}{da_1} + \frac{\partial\phi}{\partial b_2}\frac{db_2}{da_1}\right] = a_1,$$

and in like manner $p_2 = a_2$.

Again $z = b_1 x_1 + b_2 x_2 + \psi(a_1, a_2, b_1, b_2)$, and

$$q_1 = b_1 + \frac{da_1}{dx_1}\left[x_1\frac{db_1}{da_1} + x_2\frac{db_2}{da_1} + \frac{\partial\psi}{\partial a_1} + \frac{\partial\psi}{\partial a_2}\frac{da_2}{da_1} + \frac{\partial\psi}{\partial b_1}\frac{db_1}{da_1} + \frac{\partial\psi}{db_2}\frac{db_2}{da_1}\right] = b_1,$$

and similarly $q_2 = b_2$.

Hence the original differential equations are actually satisfied. If the arbitrary relation assumed—which may if convenient involve more than two of the parameters—contains two arbitrary constants, the new solution will generally be a complete primitive, since two more constants are introduced by integration.*†

* The ordinary equations to be integrated may have a singular solution with one arbitrary constant, or with none: if the arbitrary function has been chosen so as to involve three or four arbitrary constants, the whole number being thus raised to four, the solution so given may quite well be a complete primitive, and, in general, will be so.

† The above investigation in a modified form shows how to find integrals of a system of three equations

$$\left.\begin{aligned} f_1(u, v, p_1, p_2, q_1, q_2) &= 0,\\ f_2(u, v, p_1, p_2, q_1, q_2) &= 0,\\ f_3(u, v, p_1, p_2, q_1, q_2) &= 0,\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

where $u = p_1 x_1 + p_2 x_2 - y$, $v = q_1 x_1 + q_2 x_2 - z$.

One solution is to take u, v, p_1, p_2, q_1, q_2 as constants connected by the three relations (12); if they are not constants we have

$$du = x_1\,dp_1 + x_2\,dp_2,$$
$$dv = x_1\,dq_1 + x_2\,dq_2.$$

§ 29. Let us now consider the new solutions of the auxiliary equations, given by the new complete primitive. The old solutions are the six pairs of the form x_i, x_j and the six of the form u_i, u_j, where $u_1 = p_1, u_2 = p_2, u_3 = q_1, u_4 = q_2$. The bi-differentials of these twelve satisfy the relation

$$d(y, z) - p_1 d(x_1, z) - p_2 d(x_2, z) - q_1 d(y, x_1) - q_2 d(y, x_2) + (p_1 q_2 - p_2 q_1) d(x_1, x_2)$$

$$= \begin{vmatrix} x_1 + \frac{\partial\phi}{\partial p_1}, & x_2 + \frac{\partial\phi}{\partial p_2} \\ \frac{\partial\psi}{\partial p_1} & \frac{\partial\psi}{\partial p_2} \end{vmatrix} d(p_1, p_2) + \begin{vmatrix} x_1 + \frac{\partial\phi}{\partial p_1}, & \frac{\partial\phi}{\partial q_1} \\ \frac{\partial\psi}{\partial p_1}, & x_1 + \frac{\partial\psi}{\partial q_1} \end{vmatrix} d(p_1, q_1)$$

$$+ \begin{vmatrix} x_1 + \frac{\partial\phi}{\partial p_1}, & \frac{\partial\phi}{\partial q_2} \\ \frac{\partial\psi}{\partial p_1}, & x_2 + \frac{\partial\psi}{\partial q_2} \end{vmatrix} d(p_1, q_2) + \begin{vmatrix} x_1 + \frac{\partial\phi}{\partial p_2}, & \frac{\partial\phi}{\partial q_1} \\ \frac{\partial\psi}{\partial p_2}, & x_1 + \frac{\partial\psi}{\partial q_1} \end{vmatrix} d(p_2, q_1)$$

$$+ \begin{vmatrix} x_2 + \frac{\partial\phi}{\partial p_2}, & \frac{\partial\phi}{\partial q_2} \\ \frac{\partial\psi}{\partial p_2}, & x_2 + \frac{\partial\psi}{\partial q_2} \end{vmatrix} d(p_2, q_2) + \begin{vmatrix} \frac{\partial\phi}{\partial q_1}, & \frac{\partial\phi}{\partial q_2} \\ x_1 + \frac{\partial\psi}{\partial q_1}, & x_2 + \frac{\partial\psi}{\partial q_2} \end{vmatrix} d(q_1, q_2)$$

In the auxiliary equations we may take $x_1, x_2, p_1, p_2, q_1, q_2$ as independent variables, since y, z are given explicitly in terms of these six.

From (12) follow three more relations connecting the six differentials $du, dv, dp_1, dp_2, dq_1, dq_2$, so that their ratios are determinate, and therefore u, v, p_1, q_1, p_2, q_2 can only be functions of one variable. The two equations last written will then, generally, give x_1, x_2 in terms of this variable, which may not be. Hence we must have

$$du = \lambda dv, dp_1 = \lambda dq_1, dp_2 = \lambda dq_2,$$

and since $df_1 = 0$, $df_2 = 0$, $df_3 = 0$, and $du, dv, dp_1, dq_1, dp_2, dq_2$ do not vanish, λ must satisfy the equation :

$$\begin{vmatrix} \lambda\frac{df_1}{du} + \frac{df_1}{dv}, & \lambda\frac{df_1}{dp_1} + \frac{df_1}{dq_1}, & \lambda\frac{df_1}{dp_2} + \frac{df_1}{dq_2} \\ \lambda\frac{df_2}{du} + \frac{df_2}{dv}, & \lambda\frac{df_2}{dp_1} + \frac{df_2}{dq_1}, & \lambda\frac{df_2}{dp_2} + \frac{df_2}{dq_2} \\ \lambda\frac{df_3}{du} + \frac{df_3}{dv}, & \lambda\frac{df_3}{dp_1} + \frac{df_3}{dq_1}, & \lambda\frac{df_3}{dp_2} + \frac{df_3}{dq_2} \end{vmatrix} = 0.$$

If λ satisfies this equation the differential relations $du = \lambda dv, dp_1 = \lambda dq_1, dp_2 = \lambda dq_2$ reduce to two only, since u, v, p_1, q_1, p_2, q_2 are connected by the equations

$$f_1 = 0, f_2 = 0, f_3 = 0.$$

By integrating these two we find two more relations involving two arbitrary constants. Hence we may suppose v, p_1, p_2, q_1, q_2 expressed in terms of u, and find a solution by eliminating u from the following :—

$$\begin{aligned} u &= p_1 x_1 + p_2 x_2 - y, \\ v &= q_1 x_1 + q_2 x_2 - z, \\ 1 &= x_1 dp_1/du + x_2 dp_2/du. \end{aligned}$$

Then

$$d(x_1, y) = p_2 d(x_1, x_2) + \left(x_1 + \frac{\partial\phi}{\partial p_1}\right) d(x_1, p_1) + \left(x_2 + \frac{\partial\phi}{\partial p_2}\right) d(x_1, p_2) + \frac{\partial\phi}{\partial q_1} d(x_1, q_1) + \frac{\partial\phi}{\partial q_2} d(x_1, q_2),$$

and similar expressions may be found for $d(x_2, y)$, $d(x_1, z)$, $d(x_2, z)$ in terms of the bidifferentials of the pairs of independent variables.

Let c_1, c_2, c_3, c_4 be the constants of integration in a new complete primitive found by the method of §§ 27–8. Let λ be the common value of the ratios dp_1/dq_1, dp_2/dq_2, $d\phi/d\psi$. Then, after integrating the equations $dp_1/dq_1 = dp_2/dq_2 = d\phi/d\psi\ (= \lambda)$ by help of an assumed relation connecting, say, p_1, q_1, p_2, q_2, c_1, c_2 we have four relations among

$$p_1, p_2, q_1, q_2, \lambda, c_1, c_2, c_3\ c_4,$$

and we may therefore suppose p_1, p_2, q_1, q_2 expressed in terms of λ, c_1, c_2, c_3, c_4, unless λ is a constant, and therefore itself a function of c_1, c_2, c_3, c_4. Then

$$dp_1 - \lambda\, dq_1,\ dp_2 - \lambda\, dq_2,\ d\phi - \lambda\, d\psi$$

will be linear combinations of dc_1, dc_2, dc_3, dc_4, and so will some such expression as

$$\alpha\, dp_1 + \beta\, d\lambda,$$

where α vanishes if λ is one of the constants or a function of them. Conversely, dc_1, dc_2, dc_3, dc_4 will be linear combinations of

$$dp_1 - \lambda\, dq_1,\ dp_2 - \lambda\, dq_2,\ d\phi - \lambda\, d\psi,\ \alpha\, dp_1 + \beta\, d\lambda,$$

and the bidifferentials of c_1, c_2, c_3, c_4 in pairs will be linear combinations of the six following expressions :—

$$\begin{aligned}
&d(p_1, p_2) - \lambda d(q_1, p_2) - \lambda d(p_1, q_2) + \lambda^2 d(q_1, q_2),\\
&d(p_1, \phi) - \lambda d(q_1, \phi) - \lambda d(p_1, \psi) + \lambda^2 d(q_1, \psi),\\
&d(p_2, \phi) - \lambda d(q_2, \phi) - \lambda d(p_2, \psi) + \lambda^2 d(q_2, \psi),\\
&\beta d(p_1, \lambda) - \alpha\lambda d(q_1, p_1) - \beta\lambda d(q_1, \lambda),\\
&\alpha d(p_2, p_1) + \beta d(p_2, \lambda) - \alpha\lambda d(q_2, p_1) - \beta\lambda d(q_2, \lambda),\\
&\alpha d(\phi, p_1) + \beta d(\phi, \lambda) - \alpha\lambda d(\psi, p_1) - \beta\lambda d(\psi, \lambda).
\end{aligned}$$

These are combinations of the bidifferentials of p_1, p_2, q_1, q_2, in pairs, with the expressions

$$\begin{aligned}
&d(p_1, \lambda) - \lambda d(q_1, \lambda),\\
&d(p_2, \lambda) - \lambda d(q_2, \lambda),\\
&d(\phi, \lambda) - \lambda d(\psi, \lambda).
\end{aligned}$$

Now λ is a definite function of x_1, x_2, p_1, p_2, q_1, q_2, given by eliminating the differentials from the equations

$$dp_1 = \lambda dq_1,\ dp_2 = \lambda dq_2,\ d\phi = \lambda d\psi,$$
$$x_1\, dq_1 + x_2\, dq_2 + d\psi = 0.$$

By means of the first two, the third becomes

$$\left\{\lambda\frac{\partial\phi}{\partial p_1} + \frac{\partial\phi}{\partial q_1} - \lambda^2\frac{\partial\psi}{\partial p_1} - \lambda\frac{\partial\psi}{\partial q_1}\right\}dq_1 + \left\{\lambda\frac{\partial\phi}{\partial p_2} + \frac{\partial\phi}{\partial q_2} - \lambda^2\frac{\partial\psi}{\partial p_2} - \lambda\frac{\partial\psi}{\partial q_2}\right\}dq_2 = 0,$$

and the fourth

$$\left\{x_1 + \lambda\frac{\partial\psi}{\partial p_1} + \frac{\partial\psi}{\partial q_1}\right\}dq_1 + \left\{x_2 + \lambda\frac{\partial\psi}{\partial p_2} + \frac{\partial\psi}{\partial q_2}\right\}dq_2 = 0$$

The result of elimination is therefore

$$x_1\left(\lambda\frac{\partial\phi}{\partial p_2} + \frac{\partial\phi}{\partial q_2} - \lambda^2\frac{\partial\psi}{\partial p_2} - \lambda\frac{\partial\psi}{\partial q_2}\right) - x_2\left(\lambda\frac{\partial\phi}{\partial p_1} + \frac{\partial\phi}{\partial q_1} - \lambda^2\frac{\partial\psi}{\partial p_1} - \lambda\frac{\partial\psi}{\partial q_1}\right)$$
$$+\left(\lambda\frac{\partial\psi}{\partial p_1} + \frac{\partial\psi}{\partial q_1}\right)\left(\lambda\frac{\partial\phi}{\partial p_2} + \frac{\partial\phi}{\partial q_2}\right) - \left(\lambda\frac{\partial\psi}{\partial p_2} + \frac{\partial\psi}{\partial q_2}\right)\left(\lambda\frac{\partial\phi}{\partial p_1} + \frac{\partial\phi}{\partial q_1}\right) = 0.$$

This shows the form of λ as a function of x_1, x_2, p_1, p_2, q_1, q_2, not involving c_1, c_2, c_3, c_4. Now this choice of λ makes it possible to choose coefficients A, B, C, E, F, G, such that

$$x_1\, dp_1 + x_2\, dp_2 + d\phi = \mathrm{A}\,(dp_1 - \lambda dq_1) + \mathrm{B}\,(dp_2 - \lambda dq_2) + \mathrm{C}\,(d\phi - \lambda d\psi),$$
$$x_1\, dq_1 + x_2\, dq_2 + d\psi = \mathrm{E}\,(dp_1 - \lambda dq_1) + \mathrm{F}\,(dp_2 - \lambda dq_2) + \mathrm{G}\,(d\phi - \lambda d\psi).$$

Thus

$$\mathrm{A}\{d\,(p_1, \lambda) - \lambda d\,(q_1, \lambda)\} + \mathrm{B}\,\{d\,(p_2, \lambda) - \lambda d\,(q_2, \lambda)\}$$
$$+\ \mathrm{C}\{d\,(\phi, \lambda) - \lambda d\,(\psi, \lambda)\} = x_1 d\,(p_1, \lambda) + x_2 d\,(p_2, \lambda) + d\,(\phi, \lambda)$$
$$= \text{multiples of bidifferentials of } p_1, p_2, q_1, q_2$$
$$+\ \frac{\partial\lambda}{\partial x_1}\left\{x_1\, d\,(p_1, x_1) + x_2 d\,(p_2, x_1) + d\,(\phi, x_1)\right\}$$
$$+\ \frac{\partial\lambda}{\partial x_2}\left\{x_1\, d\,(p_1, x_2) + x_2 d\,(p_2, x_2) + d\,(\phi, x_2)\right\}$$
$$= \frac{\partial\lambda}{\partial x_1}\left\{d\,(y, x_1) - p_2 d\,(x_2, x_1)\right\} + \frac{\partial\lambda}{\partial x_2}\left\{d\,(y, x_2) - p_1 d\,(x_1, x_2)\right\}$$
$$+\ \text{multiples of bidifferentials of } p_1, p_2, q_1, q_2.$$

In like manner

$$\mathrm{E}\{d(p_1, \lambda) - \lambda d(q_1, \lambda)\} + \mathrm{F}\{d(p_2, \lambda) - \lambda d(q_2, \lambda)\}$$

$$+ \mathrm{G}\{d(\phi_1\lambda) - \lambda d(\psi, \lambda)\} = \frac{\partial\lambda}{\partial x_1}\{d(z, x_1) - q_2 d(x_2, x_1)\} + \frac{\partial\lambda}{\partial x_2}\{d(z, x_2) - q_1 d(x_1, x_2)\}$$

$$+ \text{ multiples of bidifferentials of } p_1, p_2, q_1, q_2.$$

Hence the three expressions

$$d(p_1, \lambda) - \lambda d(q_1, \lambda),$$
$$d(p_2, \lambda) - \lambda d(q_2, \lambda),$$
$$d(\phi, \lambda) - \lambda d(\psi, \lambda),$$

are all reduced to the same, save for a factor, by adding or subtracting multiples of the bidifferentials of x_1, x_2, x_3, x_4 and of u_1, u_2, u_3, u_4; the same is therefore true of the bidifferentials of c_1, c_2, c_3, c_4. Hence all the new complete primitives found by the method of §§ 27–8 only add one to the eleven known "bifunctionally" independent pairs of functions satisfying the auxiliary equations; one more pair, leading to a fresh complete primitive, is yet to be found.

§ 30. These results may be used to construct examples of bifunctions. For instance, the equations

$$y = p_1 x_1 + p_2 x_2 + q_1,$$
$$z = q_1 x_1 + q_2 x_2 + p_2,$$

lead to the following case among others :—

In the equations

$$\frac{dp_1}{dq_1} = \frac{dp_2}{dq_2} = \frac{dq_1}{dp_2} = \lambda,$$

put $q_2 = \lambda + a$, $dq_2 = d\lambda$, and integrate.

Thus $\quad 2p_2 = \lambda^2 + b, \quad 3q_1 = \lambda^3 + c, \quad 4p_1 = \lambda^4 + e,$

and the arbitrary constants a, b, c, e in the new solution are respectively equal to

$$q_2 - \lambda,\ 2p_2 - \lambda^2,\ 3q_1 - \lambda^3,\ 4p_1 - \lambda^4, \text{ where } \lambda^2 x_1 + x_2 + \lambda = 0.$$

Now from § 29 it follows that $d(c, e)$ can be expressed in terms of $d(a, b)$, the bidifferentials of x_1, x_2, y, z and those of p_1, p_2, q_1, q_2.

For convenience, let us write

u, v, w, x, y, z for $x_1, \lambda, p_1, p_2, q_1, q_2$ respectively; then

for x_2 we must put $-v(1 + uv)$,

for y ,, ,, $wu - xv(1 + uv) + y$,

for z ,, ,, $yu - zv(1 + uv) + x$,

so that the eight original variables connected by two equations are now expressed in terms of six.

Thus $d(3y - v^3, 4w - v^4)$ can be expressed in terms of $d(z - v, 2x - v^2)$, the six bifferentials of w, x, y, z and those of

$$u, \quad -v(1 + uv), \quad wu - xv(1 + uv) + y, \quad yu - zv(1 + uv) + x,$$

that is, of

$$u, \quad v, \quad wu - xv(1 + uv) + y, \quad yu - zv(1 + uv) + x.$$

There is no difficulty in finding the relation. It is

$$\begin{aligned} & u^2 d(3y - v^3, 4w - v^4) - 6v^2(1 + uv)^2 d(z - v, 2x - v^2) \\ & - 12u^2 d(y, w) + 12v^2(1 + uv)^2 d(z, x) \\ & + 12v^2\{(1 + uv)(y - zv^2) - u(w - xv^2)\} d(v, u) \\ & - 12v^2(1 + uv) d(v, yu - zv - uzv^2 + x) \\ & + 12uv^2 d(v, wu - xv - uxv^2 + y) = 0. \end{aligned}$$

Here then we have an identical linear relation connecting the bidifferentials of seven pairs of functions of six variables. Any one of the seven pairs is accordingly by definition a bifunction of the other six.

Second Application.

§ 31. Take now a differential equation of the second order,

$$f(x, y, z, p, q, r, s, t) = 0,$$

where p, q are the first and r, s, t the second partial derivatives of z with respect to x, y.

A complete primitive will consist of a single equation in x, y, z involving five arbitrary constants, say a_1, a_2, a_3, a_4, a_5. If we form the first and second derivatives of this equation we shall have, in all, six equations from which a_1, a_2, a_3, a_4, a_5 can be found in terms of x, y, z, p, q, r, s, t, and the original differential equation will be the result of eliminating a_1, a_2, a_3, a_4, a_5. Let u_1, u_2, u_3, u_4, u_5 represent the expressions found for a_1, a_2, a_3, a_4, a_5 respectively, in terms of x, y, z, p, q, r, s, t.

Then from the equations

$$f = 0, \quad u_1 = a_1, \quad u_2 = a_2,$$

by differentiating, we can form six equations which will involve the third derivatives of z; by eliminating these we deduce the following two differential equations to be satisfied by u_1, u_2 :—

$$\mathrm{J}(x, r, t) + p\mathrm{J}(z, r, t) + r\mathrm{J}(p, r, t) + s\mathrm{J}(q, r, t) + \mathrm{J}(y, s, t) + q\mathrm{J}(z, s, t) + s\mathrm{J}(p, s, t) + t\mathrm{J}(q, s, t) = 0, \quad \text{and}$$

$$\mathrm{J}(x, s, r) + p\mathrm{J}(z, s, r) + r\mathrm{J}(p, s, r) + s\mathrm{J}(q, s, r) + \mathrm{J}(y, t, r) + q\mathrm{J}(z, t, r) + s\mathrm{J}(p, t, r) + t\mathrm{J}(q, t, r) = 0.$$

Here J ( ) denotes the Jacobian of f, u_1, u_2 with respect to the variables specified. These equations express the conditions which are necessary and sufficient in order that

$$dz = pdx + qdy,$$
$$dp = rdx + sdy,$$
$$dq = sdx + tdy$$

may be integrable without restriction, when r, s, t are given in terms of x, y, z, p, q, by the equations

$$f = 0, u_1 = a_1, u_2 = a_2;$$

the conditions must of course be satisfied by any three of the six functions u_1, u_2, u_3, u_4, u_5, f. We thus have forty equations, of which only eight can be algebraically independent.

§ 32. The conditions to be satisfied by u_1, u_2 are linear and homogeneous in their Jacobians with respect to the eight variables x, y, z, p, q, r, s, t; of these, one is given in terms of the rest by the equation $f = 0$, and may, if convenient, be supposed not to occur in u_1, u_2: hence the auxiliary equations in this case have seven independent variables and the dependent variables do not occur explicitly: to find a solution we are therefore to form a complete bidifferential, which shall be a linear combination of the determinants of the following array :—

	$d(r, s)$,	0	$-\mathrm{X} - p\mathrm{Z} - r\mathrm{P} - s\mathrm{Q}$
	$d(r, t)$,	$\mathrm{X} + p\mathrm{Z} + r\mathrm{P} + s\mathrm{Q}$,	$-\mathrm{Y} - q\mathrm{Z} - s\mathrm{P} - t\mathrm{Q}$
	$d(s, t)$,	$\mathrm{Y} + q\mathrm{Z} + s\mathrm{P} + t\mathrm{Q}$,	0
	$d(p, r)$,	$r\mathrm{T}$,	$-r\mathrm{S} - s\mathrm{T}$
(5)	$d(p, s)$,	$s\mathrm{T}$,	$r\mathrm{R}$
	$d(p, t)$,	$-r\mathrm{R} - s\mathrm{S}$	$s\mathrm{R}$
	$d(q, r)$,	$s\mathrm{T}$,	$-s\mathrm{S} - t\mathrm{T}$
	$d(q, s)$,	$t\mathrm{T}$,	$s\mathrm{R}$
	$d(q, t)$,	$-s\mathrm{R} - t\mathrm{S}$,	$t\mathrm{R}$
(10)	$d(z, r)$,	$p\mathrm{T}$,	$-p\mathrm{S} - q\mathrm{T}$
	$d(z, s)$,	$q\mathrm{T}$,	$p\mathrm{R}$
	$d(z, t)$,	$-p\mathrm{R} - q\mathrm{S}$,	$q\mathrm{R}$
	$d(x, r)$,	T,	$-\mathrm{S}$
	$d(x, s)$,	0,	R

(15)	$d(x, t)$,	$-\mathrm{R}$	0
	$d(y, r)$,	0,	$-\mathrm{T}$
	$d(y, s)$,	T,	0
	$d(y, t)$,	$-\mathrm{S}$,	R
	$d(x, p)$,	0,	0
(20)	$d(y, p)$,	0,	0
	$d(z, p)$,	0,	0
	$d(x, q)$,	0,	0
	$d(y, q)$,	0,	0
	$d(z, q)$,	0,	0
(25)	$d(p, q)$,	0,	0
	$d(x, z)$,	0,	0
	$d(y, z)$,	0,	0
	$d(x, y)$,	0,	0

X, P . . . are written for $\partial f/\partial x$, $\partial f/\partial p$ . . .

Of these twenty-eight rows, only twenty-one are independent. For instance, multiply the 1st, 2nd, 4th, 7th, 10th, 13th, 16th by $-$ S, $-$ T, P, Q, Z, X, Y respectively and add; the resulting row is

$$d(f, r), \quad 0, \quad 0,$$

which vanishes since $f = 0$ by hypothesis.

Suppose $d(u_1, u_2)$ to be the complete bidifferential formed from the determinants of the array, then to complete the solution we have to find r, s, t from the equations

$$f = 0, \quad u_1 = a_1, \quad u_2 = a_2,$$

and integrate the equations

$$dz = pdx + qdy, \quad dp = rdx + sdy, \quad dq = sdx + tdy.$$

It will amount to the same thing if we treat u_1 as known in the auxiliary equations. They must be satisfied if u_3, u_4, u_5 are substituted in turn for u_2. Now two homogeneous linear partial differential equations in seven independent variables can at most have five common solutions, and here one of these, u_1, is known; the other four may be taken as u_2, u_3, u_4, u_5.

§ 33. Any two of the five functions x, y, z, p, q will satisfy the auxiliary equations, but as we have to solve for r, s, t, these solutions will not serve our purpose. They are ten in number, and ten more will be given by taking in pairs the expressions u_1, u_2, u_3, u_4, u_5 given by any complete primitive. These twenty are not all bifunctionally independent, for since there are three relations* among the ten expressions

$$x, y, z, p, q, u_1, u_2, u_3, u_4, u_5,$$

* Compare § 34, p. 184.

three linear relations can be formed connecting the twenty bidifferentials; one is formed from each pair of equations as at § 17 (8). Hence seventeen bifunctionally independent solutions of the auxiliary equations are known when we have one complete primitive. The full number is nineteen $\left(\frac{7.6}{1.2} - 2\right)$, and in order to know all we must have one, or possibly two (see § 41, p. 190), more complete primitives.

§ 34. New solutions found by varying the parameters may be divided into two classes, according as the parameters are or are not all functions of one variable; solutions of the former class only occur in exceptional cases, and the principles of § 21 apply to them with slight modification.

Let the three equations connecting

$$x,\ y,\ z,\ p,\ q,\ u_1,\ u_2,\ u_3,\ u_4,\ u_5$$

be
$$\phi_i(x,\ y,\ z,\ p,\ q,\ u_1,\ u_2,\ u_3,\ u_4,\ u_5) = 0\ (i = 1, 2, 3);$$

(the forms ϕ_1, ϕ_2, ϕ_3 are not unrestricted, but must be such that the following relations hold identically

$$\frac{\partial(\phi_1, \phi_2, \phi_3)}{\partial(x, p, q)} + p\frac{\partial(\phi_1, \phi_2, \phi_3)}{\partial(z, p, q)} = 0,$$

$$\frac{\partial(\phi_1, \phi_2, \phi_3)}{\partial(y, p, q)} + q\frac{\partial(\phi_1, \phi_2, \phi_3)}{\partial(z, p, q)} = 0;$$

or we may take ϕ_1 as not involving p, q and ϕ_2 as $p\,\partial\phi_1/\partial z + \partial\phi_1/\partial x$

ϕ_3 as $q\,\partial\phi_1/\partial z + \partial\phi_1/\partial y$),

then the variations of the parameters must satisfy the three equations

$$\sum_{r=1}^{r=5} \frac{\partial\phi_i}{\partial u_r}\,du_r = 0\ (i = 1,\ 2,\ 3),$$

in order that the same relations may subsist among x, y, z, p, q, r, s, t and the parameters, as held when the parameters were constant.

If the parameters are functions of one variable, their forms must be so chosen that the three equations last written reduce to one only, otherwise we shall have five relations connecting x, y, z, p, q with this single variable.

§ 35. If the parameters are not functions of one variable, only the equations

$$\sum_{r=1}^{r=5} \frac{\partial\phi_i}{\partial u_r}\,du_r = 0$$

are equivalent to six, and determine the partial derivatives of u_3, u_4, u_5 with respect to u_1, u_2 in terms of the five parameters and x, y, z, p, q. By help of the relations $\phi_i = 0$ we may suppose x, y, z, p, q eliminated and thus arrive at a system of four partial differential equations connecting u_1, u_2, u_3, u_4, u_5.

The original system may also be taken to consist of four equations connecting five variables x, y, z, p, q, namely:

$$dz/dx = p,\ dz/dy = q,\ dp/dy = dq/dx$$
$$f(x, y, z, p, q, dp/dx, dp/dy, dq/dy) = 0,$$

and so the method of variation of parameters does not lead to any simplification of the problem in general.

§ 36. The interchange of variables and parameters is again possible; it is, perhaps, made clearer by taking three equations of perfectly general form,

$$\phi_i\ (x_1, x_2, x_3, x_4, x_5, u_1, u_2\ u_3, u_4, u_5) = 0\ (i = 1, 2, 3),$$

connecting two sets, each of five quantities.

Whichever set we suppose constant and eliminated by differentiation, we are led to a system of four partial differential equations connecting the quantities of the other set, two of the five being taken as independent variables. A new solution of either of these systems of differential equations will in general yield a new solution of the other.

Suppose, for instance, that we have a new solution of the u equations; this gives u_3, u_4, u_5, say, in terms of u_1, u_2. Then the six equations included in

$$\sum_{r=1}^{r=5} \frac{\partial\phi_i}{\partial u_r}\, du_r = 0\ (i = 1, 2, 3)$$

give two relations among $x_1, \ldots u_1, u_2$, since the four differential equations, which are consequences of these six, are supposed satisfied; by the help of these two, u_1, u_2, may be eliminated from the three relations $\phi_1 = 0$, $\phi_2 = 0$, $\phi_3 = 0$, and thus three relations are given connecting x_1, x_2, x_3, x_4, x_5; these three will constitute a solution of the x system of differential equations.

§ 37. In this more general case there will not seemingly, as a rule, be any more solutions for either system of differential equations. For the derivatives, say, of x_3, x_4, x_5 with respect to x_1, x_2 are given in terms of these five variables and two others, say u_1, u_2. The forms we may assign to u_1, u_2 are then restricted by three differential equations derived from the three conditions

$$\frac{d^2x_r}{dx_1\, dx_2} = \frac{d^2x_r}{dx_2\, dx_1}\ (r = 3, 4, 5),$$

and thus, generally speaking, no forms of u_1, u_2 will be suitable. In some cases the conditions are not inconsistent, and we may form an array by the method of § 11 such that if $d(\theta,\chi)$ is a combination of its determinants, then $\theta = a$, $\chi = b$, $\phi_1 = 0$,

$\phi_2 = 0$, $\phi_3 = 0$ will give suitable values for u_1, u_2. This array will have four columns and forty-five rows, ten such as

$$d(x_i, x_j),\ 0,\ 0,\ 0,$$

ten such as

$$d(u_i, u_j),\ 0,\ 0,\ 0,$$

and twenty-five of the following type. In the first column there is $d(x_i, u_j)$, in the $(r+1)$th the minor of $\partial^2\phi_r/\partial x_i \partial u_j$, in the determinant

$$\begin{vmatrix} \frac{\partial^2\phi_r}{\partial x_1\,\partial u_1}, & \frac{\partial^2\phi_r}{\partial x_2\,\partial u_1}, & \frac{\partial^2\phi_r}{\partial x_3\,\partial u_1}, & \frac{\partial^2\phi_r}{\partial x_4\,\partial u_1}, & \frac{\partial^2\phi_r}{\partial x_5\,\partial u_1} & \frac{\partial\phi_1}{\partial u_1} & \frac{\partial\phi_2}{\partial u_1} & \frac{\partial\phi_3}{\partial u_1} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \frac{\partial\phi_1}{\partial u_2} & \frac{\partial\phi_2}{\partial u_2} & \frac{\partial\phi_3}{\partial u_2} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \frac{\partial\phi_1}{\partial u_3} & \frac{\partial\phi_2}{\partial u_3} & \frac{\partial\phi_3}{\partial u_3} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \frac{\partial\phi_1}{\partial u_4} & \frac{\partial\phi_2}{\partial u_4} & \frac{\partial\phi_3}{\partial u_4} \\ \frac{\partial^2\phi_r}{\partial x_1\,\partial u_5} & \cdot & \cdot & \cdot & \frac{\partial^2\phi_r}{\partial x_5\,\partial u_5} & \frac{\partial\phi_1}{\partial u_5} & \frac{\partial\phi_2}{\partial u_5} & \frac{\partial\phi_3}{\partial u_5} \\ \frac{\partial\phi_1}{\partial x_1} & \frac{\partial\phi_1}{\partial x_2} & \frac{\partial\phi_1}{\partial x_3} & \frac{\partial\phi_1}{\partial x_4} & \frac{\partial\phi_1}{\partial x_5} & 0 & 0 & 0 \\ \frac{\partial\phi_2}{\partial x_1} & \frac{\partial\phi_2}{\partial x_2} & \frac{\partial\phi_2}{\partial x_3} & \frac{\partial\phi_2}{\partial x_4} & \frac{\partial\phi_2}{\partial x_5} & 0 & 0 & 0 \\ \frac{\partial\phi_3}{\partial x_1} & \frac{\partial\phi_3}{\partial x_2} & \frac{\partial\phi_3}{\partial x_3} & \frac{\partial\phi_3}{\partial x_4} & \frac{\partial\phi_3}{\partial x_5} & 0 & 0 & 0 \end{vmatrix}$$

for

$$r = 1,\ 2,\ 3.$$

This interchange of variables and parameters may take place whenever their numbers are equal, the differential equations being of the first degree.

Examples.

§ 38. I. As an example of the method of solution take the equation $\alpha = \beta$, where α is a function of r, s, $p - sy$, x and β a function of s, t, $q - sx$, y.

In the array (§ 32) multiply the first row by $\partial\alpha/\partial r$, the fifth by $\partial\alpha/\partial p$, the fourteenth by $\partial\alpha/\partial x$, the seventeenth by $-s\,\partial\alpha/\partial p$, and add; the resulting row is

$$d(\alpha, s),\ 0,\ 0.$$

Hence we take $\alpha = \beta = a$, $s = b$,

$$z = bxy + X + Y,$$

X being a function of x only and Y a function of y only. Then $\alpha = a$ is a relation connecting x, dX/dx, d^2X/dx^2, and $\beta = a$ is a relation connecting y, dY/dy, d^2Y/dy^2, and by solving these for X, Y respectively we shall have the complete primitive.

§ 39. II. As a second example take the equation

$$F(r, s, t, p - sy, q - ty, z - qy + \tfrac{1}{2}ty^2, x) = 0.$$

Here the third row of the array is

$$d(s, t),\ 0,\ 0,$$

so that the functions s, t satisfy the auxiliary equations. Put, then, $s = a$, $t = b$; thus

$$\begin{aligned} q &= ax + by + c \\ z &= axy + \tfrac{1}{2}by^2 + cy + X, \end{aligned}$$

the last term being a function of x only. The differential equation thus becomes

$$F(d^2X/dx^2,\ a,\ b,\ dX/dx,\ ax + c,\ X,\ x) = 0,$$

an ordinary equation of the second order giving X in terms of x and two more arbitrary constants; hence the finding of a complete primitive is reduced to the solution of the equation last written.

§ 40. III. If the equation is of the particular form $F(r, s, t, p - rx - sy, q - sx - ty, z - px - qy + \frac{1}{2}rx^2 + sxy + \frac{1}{2}ty^2) = 0$, the first three rows of the array are

$$\begin{array}{ccc} d(r, s) & 0 & 0 \\ d(r, t) & 0 & 0 \\ d(s, t) & 0 & 0. \end{array}$$

Hence any two of the three functions r, s, t will satisfy the auxiliary equations, and a complete primitive is given by putting

$$r = a, \quad s = h, \quad t = b.$$

Hence $p = ax + hy + g$, $q = hx + by + f$

$$z = c + gx + fy + \tfrac{1}{2}(ax^2 + 2hxy + by^2),$$

where a, b, c, f, g, h are constants satisfying the relation

$$F(a, h, b, g, f, c) = 0.$$

This is a case in which other solutions are readily given by supposing the parameters variable and functions of one variable only, say a. The variations must satisfy the conditions

$$x^2da + 2xy\,dh + y^2db + 2x\,dg + 2y\,df + 2dc = 0,$$
$$x\,da + y\,dh + dg = 0,\ x\,dh + y\,db + df = 0,$$

whence follows $xdg+ydf+2dc = 0$, a simpler relation that may be taken instead of the first of the three.

These equations will define x, y in terms of the single variable a, unless all the first minors of

$$\begin{vmatrix} da & dh & dg \\ dh & db & df \\ dg & df & 2dc \end{vmatrix} \text{ vanish.}$$

We thus have three ordinary differential equations connecting a, b, c, f, g, h; they are connected also by the relation $F(a, h, b, g, f, c) = 0$, and the fifth relation among them may be chosen arbitrarily, so that we may put $h = \phi(a)$, an arbitrary function.

Then we have

$$db/da = \{\phi'(a)\}^2, \quad df/da = \phi'(a)\, dg/da,$$

$$2dc/da = (dg/da)^2,$$

$$F(a, \phi(a), b, g, f, c) = 0$$

as the equations determining b, g, f, c in terms of a. These are to be integrated, and then a is to be eliminated from the equations

$$x + y\, dh/da + dg/da = 0,$$

$$z = c + gx + fy + \tfrac{1}{2}(ax^2 + 2hxy + by^2).$$

The result of elimination will be a solution of the differential equation. Three constants are introduced by integration, and thus, if the function ϕ involves two constants, the new solution will generally be a complete primitive.

§ 41. The new complete primitive gives new solutions of the auxiliary equations which we shall now examine. Let α_1, α_2, α_3, α_4, α_5 be the new set of parameters. Then a, h, g, b, c, f are connected with these parameters by five equations, one of which is the original equation $F = 0$. These five relations are such, that if

$$dh = \lambda da, \ dg = \mu da,$$

then $$db = \lambda^2 da, \ 2dc = \mu^2 da, \ df = \lambda\mu da;$$

of these five, the first two define λ, μ in terms of a, α_1, α_2, α_3, α_4, α_5, and the others must then follow from the five equations that give h, g, b, c, f in terms of a and the same new constants. Thus, in general, we may suppose a, h, g, b, c, f, μ, expressed in terms of λ, α_1, α_2, α_3, α_4, α_5 and the expressions will be such that

$$dh - \lambda da, \ dg - \mu da, \ db - \lambda^2 da, \ 2dc - \mu^2 da, \ df - \lambda\mu da$$

involve only the differentials of $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$. One of these five is expressible in terms of the other four, since

$$\frac{\partial F}{\partial a}da + \frac{\partial F}{\partial h}dh + \frac{\partial F}{\partial b}db + \frac{\partial F}{\partial g}dg + \frac{\partial F}{\partial f}df + \frac{\partial F}{\partial c}dc = 0,$$

while one of the relations connecting $\lambda, \mu, a, b, \ldots$ is

$$\frac{\partial F}{\partial a} + \lambda\frac{\partial F}{\partial h} + \mu\frac{\partial F}{\partial g} + \lambda^2\frac{\partial F}{\partial b} + \lambda\mu\frac{\partial F}{\partial f} + \tfrac{1}{2}\mu^2\frac{\partial F}{\partial c} = 0.$$

Some expression such as $\nu d\lambda - \rho da$ will also involve the differentials of $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$ only. Hence the differentials of $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$ will be linear combinations of $\nu d\lambda - \rho da$, $dh - \lambda da$, $dg - \mu da$, $db - \lambda^2 da$, $df - \lambda\mu da$, $2dc - \mu^2 da$, of which the last five satisfy a linear relation.

Thus the bidifferentials of $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$ in pairs will be linear combinations of the bidifferentials of a, b, c, f, g, h (only five of the six need be used) in pairs, and of the expressions

$$d(h, \lambda) - \lambda d(a, \lambda),\ d(g, \lambda) - \mu d(a, \lambda),\ d(b, \lambda) - \lambda^2 d(a, \lambda),$$
$$d(f, \lambda) - \lambda\mu d(a, \lambda),\ 2d(c, \lambda) - \mu^2 d(a, \lambda),$$

of which last five, only four are independent.

Now λ, μ are connected not only by the equation

$$\frac{\partial F}{\partial a} + \lambda\frac{\partial F}{\partial h} + \mu\frac{\partial F}{\partial g} + \lambda^2\frac{\partial F}{\partial b} + \lambda\mu\frac{\partial F}{\partial f} + \tfrac{1}{2}\mu^2\frac{\partial F}{\partial c} = 0,$$

but also by the equation

$$x + \lambda y + \mu = 0,$$

so that they are definite functions of x, y, a, b, f, g, h.

Again
$$p = ax + hy + g,$$
$$d(p, x) - hd(y, x) = xd(a, x) + yd(h, x) + d(g, x),$$
$$d(p, y) - ad(x, y) = xd(a, y) + yd(h, y) + d(g, y).$$

Thus
$$\begin{aligned} & y\{d(h, \lambda) - \lambda d(a, \lambda)\} + \{d(g, \lambda) - \mu d(a, \lambda)\} \\ &= xd(a, \lambda) + yd(h, \lambda) + d(g, \lambda) \\ &= \frac{\partial\lambda}{\partial x}[d(p, x) - hd(y, x)] + \frac{\partial\lambda}{\partial y}[d(p, y) - ad(x, y)], \end{aligned}$$

+ multiples of bidifferentials of a, b, c, f, g, h.

In like manner

$$y\{d(b, \lambda) - \lambda^2 d(a, \lambda)\} + \{d(f, \lambda) - \lambda\mu d(a, \lambda)\} + x\{d(h, \lambda) - \lambda d(a, \lambda)\}$$
$$= xd(h, \lambda) + yd(b, \lambda) + d(f, \lambda)$$
$$= \frac{\partial\lambda}{\partial x}[d(q, x) - bd(y, x)] + \frac{\partial\lambda}{\partial y}[d(q, y) - hd(x, y)],$$

+ multiples of bidifferentials of a, b, c, f, g, h.

Lastly,

$$2xy\{d(h, \lambda) - \lambda d(a, \lambda)\} + y^2\{d(b, \lambda) - \lambda^2 d(a, \lambda)\} + 2x\{d(g, \lambda) - \mu d(a, \lambda)\}$$
$$+ 2y\{d(f, \lambda) - \lambda\mu d(a, \lambda)\} + \{2d(c, \lambda) - \mu^2 d(a, \lambda)\}$$
$$= x^2 d(a, \lambda) + 2xyd(h, \lambda) + y^2 d(b, \lambda) + 2xd(g, \lambda) + 2yd(f, \lambda) + 2d(c, \lambda)$$
$$= 2\{d(z, x) - (hx + by + f)d(y, x)\}\partial\lambda/dx$$
$$+ 2\{d(z, y) - (ax + hy + g)d(x, y)\}\partial\lambda/dy,$$

+ multiples of bidifferentials of a, b, c, f, g, h.

Hence, in all, nine combinations of the ten bidifferentials of $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$ can be expressed in terms of the bidifferentials of x, y, z, p, q and of a, b, c, f, g, h; that is, in terms of the bidifferentials of the seventeen known independent pairs of functions satisfying the auxiliary equations: thus the new complete primitive adds only one to the number of these known bifunctionally independent pairs, and one more must be added in order to give the full number.

This theory enables us again to construct examples of bifunctions of a number of known pairs which may reach eighteen.

§ 42. The foregoing investigation may be modified so as to give singular solutions of a pair of differential equations of the form in question, say

$$F_1(r, s, t, \bar{p}, \bar{q}, \bar{z}) = 0,$$
$$F_2(r, s, t, \bar{p}, \bar{q}, \bar{z}) = 0,$$

where

$$\bar{p} = p - rx - sy,$$
$$\bar{q} = q - sx - ty,$$
$$\bar{z} = z - \tfrac{1}{2}(p + \bar{p})x - \tfrac{1}{2}(q + \bar{q})y.$$

A complete primitive would be given by supposing $r, s, t, \bar{p}, \bar{q}, \bar{z}$ constants connected by the above equations. Another solution would be given by solving the total differential equations found by supposing the relations

$$x\,dr + y\ \ ds + d\bar{p} = 0,$$
$$x\,ds + y\ \ dt + d\bar{q} = 0,$$
$$x\,d\bar{p} + y\ \ d\bar{q} + 2d\bar{z} = 0,$$

to reduce to the same relation linear in x and y. That is, we must solve the system

$$ds = \lambda dr, \quad dt = \lambda^2 dr, \quad d\bar{p} = \mu dr,$$

where λ, μ are given in terms of $\bar{p}$, $\bar{q}$, $\bar{z}$, r, s, t by the relations

$$\frac{\partial F_1}{\partial r} + \lambda\frac{\partial F_1}{\partial s} + \lambda^2\frac{\partial F_1}{\partial t} + \mu\frac{\partial F_1}{\partial \bar{p}} + \lambda\mu\frac{\partial F_1}{\partial \bar{q}} + \tfrac{1}{2}\mu^2\frac{\partial F_1}{\partial \bar{z}} = 0,$$
$$\frac{\partial F_2}{\partial r} + \lambda\frac{\partial F_2}{\partial s} + \lambda^2\frac{\partial F_2}{\partial t} + \mu\frac{\partial F_2}{\partial \bar{p}} + \lambda\mu\frac{\partial F_2}{\partial \bar{q}} + \tfrac{1}{2}\mu^2\frac{\partial F_2}{\partial \bar{z}} = 0,$$

and $\bar{q}$, $\bar{z}$ in terms of $\bar{p}$, r, s, t by the relations $F_1 = 0$, $F_2 = 0$.

The complete primitive of these ordinary equations will involve three arbitrary constants, and there may be singular solutions with a lower number; none of these will therefore constitute a complete primitive of the partial differential system

$$F_1 = 0, \ F_2 = 0.$$